Master of Science (Mathematics) (DDE)

Semester – II

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MEASURE AND INTEGRATION THEORY

DIRECTORATE OF DISTANCE EDUCATION

MAHARSHI DAYANAND UNIVERSITY, ROHTAK

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MASTER OF SCIENCE (MATHEMATICS)
Measure and Integration Theory (Semester–II)
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M. Marks = 100
Term End Examination = 80
Time = 3 hrs Assignment = 20

Course Outcomes

Students would be able to:

CO1 Describe the shortcomings of Riemann integral and benefits of Lebesgue integral.

CO2 Understand the fundamental concept of measure and Lebesgue measure.

CO3 Learn about the differentiation of monotonic function, indefinite integral, use of the fundamental theorem of calculus.

Section - I

Set functions, Intuitive idea of measure, Elementary properties of measure, Measurable sets and their fundamental properties. Lebesgue measure of a set of real numbers, Algebra of measurable sets, Borel set, Equivalent formulation of measurable sets in terms of open, Closed, F_σ and G_δ sets, Non measurable sets.

Section - II

Measurable functions and their equivalent formulations. Properties of measurable functions. Approximation of a measurable function by a sequence of simple functions, Measurable functions as nearly continuous functions, Egoroff theorem, Lusin theorem, Convergence in measure and F. Riesz theorem. Almost uniform convergence.

Section - III

Shortcomings of Riemann Integral, Lebesgue Integral of a bounded function over a set of finite measure and its properties. Lebesgue integral as a generalization of Riemann integral, Bounded convergence theorem, Lebesgue theorem regarding points of discontinuities of Riemann integrable functions, Integral of non-negative functions, Fatou Lemma, Monotone convergence theorem, General Lebesgue Integral, Lebesgue convergence theorem.

Section - IV

Vitali covering lemma, Differentiation of monotonic functions, Function of bounded variation and its representation as difference of monotonic functions, Differentiation of indefinite integral, Fundamental theorem of calculus, Absolutely continuous functions and their properties.

Note : The question paper of each course will consist of **five** Sections. Each of the sections **I to IV** will contain **two** questions and the students shall be asked to attempt **one** question from each. **Section-V** shall be **compulsory** and will contain **eight** short answer type questions without any internal choice covering the entire syllabus.

Books Recommended :

1. Walter Rudin, Principles of Mathematical Analysis (3rd edition) McGraw-Hill, Kogakusha, 1976, International Student Edition.
2. H.L. Royden, Real Analysis, Macmillan Pub. Co., Inc. 4th Edition, New York, 1993.
3. P. K. Jain and V. P. Gupta, Lebesgue Measure and Integration, New Age International (P) Limited Published, New Delhi, 1986.
4. G.De Barra, Measure Theory and Integration, Wiley Eastern Ltd., 1981.
5. R.R. Goldberg, Methods of Real Analysis, Oxford & IBH Pub. Co. Pvt. Ltd, 1976.
6. R. G. Bartle, The Elements of Real Analysis, Wiley International Edition, 2011.

Contents

SECTION – I

MEASURABLE SETS

Introduction

In measure theory, a branch of mathematics, the concept of **Lebesgue measure**, was given by French mathematician Henri Lebesgue in 1901. Sets that can be assigned a Lebesgue measure are called **Lebesgue-measurable**; the measure of the Lebesgue-measurable set A is here denoted by $m^*(A)$.

Lebesgue Measure

In this section we shall define Lebesgue Measure, which is a generalization of the idea of length.

1.1 Definition. The length l(I) of an interval I with end points a and b is defined as the difference of the end points. In symbols, we write.

$$l\,(I) \;=\; b - a.$$

1.2 Definition. A function whose domain of definition is a class of sets is called a Set Function. For example, length is a set function. The domain being the collection of all intervals.

1.3 Definition. An extended real – valued set function μ defined on a class E of sets is called Additive if $A \in E, B \in E, A \cup B \in E \text{ and } A \cap B = \phi, imply$

$$\mu\,(A \cup B) \;=\; \mu\,(A) \;+\; \mu\,(B)$$

1.4 Definition. An extended real valued set function μ defined on a class E of sets is called finitely additive if for every finite disjoint classes $\{A_1, A_2, \ldots, A_n\}$of sets in E, whose union is also in E,

we have

$$\mu(U_{i=1}^{n} A_i\,) \;=\; \sum_{i=1}^{n} \mu(A_i)$$

1.5 Definition. An extended real–valued set function μ defined on a class E of sets is called countably additive it for every disjoint sequence $\{A_n\}$ of sets in E whose union is also in E, we have

$$\mu(U_{i=1}^{\infty} A_i\,) \;=\; \sum_{i=1}^{\infty} \mu(A_i)$$

1.6 Definition. Length of an open set is defined to be the sum of lengths of the open intervals of which it is composed of. Thus, if G is an open set, then

$$l(G) \;=\; \sum_{n} l(I_n)$$

where

$$G \;=\; U_n\, I_n\,, I_{n_1} \cap I_{n_2} \;=\; \phi \;if\; n_1 \neq n_2.$$

1.7 Definition. The Lebesgue Outer Measure or simply the outer measure m* of a set A is defined as

$$m^*(A) = \inf_{A \subseteq UI_n} \sum l(I_n).$$

where the infimum is taken over all finite or countable collections of intervals $\{I_n\}$ such that $A \subseteq UI_n$

Since the lengths are positive numbers, it follows from the definition of m^* that $m^*(A) \geq 0$.

1.8 Remark: (i) If $A \subseteq B$,then $m^*(A) \leq m^*(B)$ i.e. outer-measure has monotone property.

Proof: By definition of outer-measure, for each $\varepsilon > 0$, there exist a countable collection of open interval $\{I_n\}$ such that $B \subseteq U_n I_n$ and

$m^*(B) + \varepsilon > \sum_n l(I_n)$ ...1)

now $A \subseteq B$ *and* $B \subseteq U_n I_n$

$\Rightarrow A \subseteq U_n I_n$

$m^*(A) \leq \sum_n l(I_n)$

$< m^*(B) + \varepsilon$ (*using* 1))

⇨ $m^*(A) < m^*(B) + \varepsilon$

but $\varepsilon > 0$ is arbitrary, $m^*(A) \leq m^*(B)$ hence proved.

(ii) Outer-measure of a set is always non-negative.

1.9 Theorem. Outer measure is translation invariant.

Proof. Let $\epsilon > 0$ be given. Then by definition of outer measure, There exist a countable collection of intervals $\{I_n\}$ such that $A \subseteq \cup I_n$ *and*

$m^*(A) + \epsilon > \sum_n l(I_n)$.

Now, $A \subseteq \bigcup_n (I_n)$

$\Rightarrow A + x \subseteq \bigcup_n (I_n + x)$,

$\Rightarrow m^*(A + x) \leq \sum_n l(I_n + x) = \Sigma\, l(I_n)$ [length is translation invariant]

$\leq m^* A + \epsilon$

Since ϵ is arbitrary positive number, we have

(2) $m^*(A + x) \leq m^*(A)$ (1)

To prove reverse inequality, Let $\epsilon > 0$ be given. Then by definition of outer measure, There exist a countable collection of intervals $\{J_n\}$ such that

$A + x \subseteq \bigcup_n J_n$ *and*

$m^*(A + x) + \epsilon > \sum_n l(J_n)$.

Now, $A + x \subseteq \bigcup_n J_n$

⇨ $A \subseteq \bigcup_n (J_n - x)$
⇨ $m^* (A) \leq \sum_n \ l(J_n - x)$
⇨ $m^* (A) \leq \sum_n \ l(J_n) < m^* (A + x) \ + \ \in$
⇨ $m^* (A) \leq m^*(A \ + \ x)$ (2)

Then Combining (1) and (2), the required result follows.

i.e., $m^* (A) = \ m^*(A + x)$

1.10 Theorem. The outer measure of an interval is its length.

Proof. CASE (1) Let us suppose, first I is a closed and bounded interval, say I = [a, b]

To prove: m*(I) = ℓ [a, b] =b - a.

Now for each ε> 0, I = [a, b] ⊆ (a - ε, b+ ε) then

by definition of outer-measure

=> m*(I) ≤ ℓ (a - ε, b+ ε) ≤ (b+ ε- a+ ε)

=> m*(I) ≤ b-a + 2 ε

since ε is an arbitrary, m*(I) ≤ b-a = ℓ (I) (1)

Now to prove, m*(I) = b-a, then it is sufficient to prove m*(I) ≥ b-a. let $\{I_n\}$ be a countable collection of open intervals which covering I i.e.

I ⊆ $\bigcup_n I_n$

$\sum_n \ell(I_n)$ ≥ b-a for all n ∈ N so it is sufficient to prove that

inf $\sum_n \ell(I_n)$ ≥ b-a

since I = [a, b] is compact, then by Heine Boral theorem, we can select a finite number of open intervals from this $\{I_n\}$ such that their union contains I.

Let the intervals be J_1, J_2, ..., Jp such that $\bigcup_{i=1}^{p} J_i$ ⊇ [a, b].

Now it is sufficient to prove $\sum_{i=1}^{p} \ell(J_i)$ ≥ b-a (2)

Now a ∈ I = [a, b], there exist open interval $J_1 = (a_1, b_1)$ from the above-mentioned finite no. of intervals such that $a_1 < a \leq b$ then $b_1 \in I$.

Again, there exist an open interval (a_2, b_2) from the finite collection J_1, J_2, ..., Jp such that $a_2 < b_1 < b_2$. Continuing this, we get a sequence of open intervals

(a_1, b_1), (a_2, b_2), ..., (a_p, b_p) from J_1, J_2, ..., Jp satisfying $a_i < b_{i-1} <$ bi ,i= 2,3,.....,p since the collection is finite so the process must stop with an interval satisfying $a_p < b_{p-1} < b_p$ and $a_p < b < b_p$

$\sum_n \ell(I_n) \ \geq \sum_{i=1}^{p} \ell(J_i) = \ell\ (a_1, b_1) + \ell\ (a_2, b_2) + ... \ \ell\ (a_p, b_p)$

$= (b_1 - a_1) + (b_2 - a_2) + ... + (b_p - a_p)$

$= b_p + (b_{p-1} - a_p) + ... + b_1 - a_2 - a_1$

$> b_p - a_1$

$> b-a$

$\Rightarrow \inf \sum_n \ell(I_n) \geq b-a$

$\Rightarrow m^*(I) \geq b-a$ (4)

Hence result is proved in the case when I closed and bounded interval.

CASE (2) let I be bounded open interval with end points a and b, then for every real no. $\varepsilon > 0$ $[a+\varepsilon, b-\varepsilon] \subset I \subset [a, b]$

$\Rightarrow m^*[a+\varepsilon, b-\varepsilon] \leq m^*(I) \leq m^*[a, b]$

$\Rightarrow \ell[a+\varepsilon, b-\varepsilon] \leq m^*(I) \leq \ell[a, b]$ (by case 1)

$\Rightarrow b-\varepsilon-a-\varepsilon \leq m^*(I) \leq b-a$

since ε is arbitrary,

we get $b-a \leq m^*(I) \leq b-a$

$\Rightarrow m^*(I) = b-a.$

CASE (3) if I is the unbounded interval, then for each real no. $r > 0$, we can find bounded closed interval $J \subset I$ such that $\ell(J) > r$

Now $J \subset I \Rightarrow m^*(J) \leq m^*(I)$

$\Rightarrow \ell(J) \leq m^*(I)$

$\Rightarrow m^*(I) > r$ since this hold for each real no. r,

we get $m^*(I) = \infty = \ell(I)$

i.e. outer-measure is of an interval equal to its length.

1.11 Theorem. Let $\{A_n\}$ be a countable collection of sets of real numbers. Then $m^*(\cup A_n) \leq \Sigma\, m^* A_n$.

Proof. Proof. If one of the sets A_n has infinite outer measure, the inequality holds trivially. So suppose $m^*\{A_n\}$ is finite. Then, given $\epsilon > 0$, there exists a countable collection $\{I_{n,i}\}$ of open intervals such that $A_n \subset U_i\, I_{n,i}$ and

$$\Sigma_i\ l(I_{n,i}) < m^*(A_n) + \frac{\varepsilon}{2^n}$$

by the definition of $m^*\{A_n\}$.

Now the collection $[I_{n,i}]_{n,i} = U_n\, [I_{n,i}]_i$ is countable, being the union of a countable number of countable collections, and covers $\bigcup_n A_n$. Thus

$$m^*\left(\bigcup_n A_n\right) \leq \Sigma_{n,i}\, l(I_{n,i})$$

$$= \Sigma_n \Sigma_i l(I_n, i)$$

$$< \left(m^*(A_n) + \frac{\epsilon}{2^n} \right)$$

$$= \Sigma_n m^* A_n + \Sigma_n \frac{\epsilon}{2^n}$$

$$= \Sigma_n m^* A_n + \epsilon \Sigma_n \frac{1}{2^n}$$

$$= \Sigma\, m^* A_n + \epsilon$$

Since ϵ is an arbitrary positive number, it follows that

$m^*(\bigcup_n A_n) \leq \Sigma\, m^*(A_n)$.

1.12 Theorem. Outer-measure of singleton set of reals is zero

Proof: Let $A = \{a\}$ Then, since $A = \{a\}, \{a\} \subseteq \left(a - \frac{1}{n}, a + \frac{1}{n}\right) \forall n \in N$

⇨ $m^*(a) \leq m * \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$

⇨ $m^*(a) \leq \frac{2}{n}$

⇨ $0 \leq m^*(a) \leq \frac{2}{n}$ for each n.

In limiting case $m^*(a) = 0$.

1.13 Theorem. Outer-measure of null set is zero.

Proof: Since $\phi \subseteq \left(-\frac{1}{n}, \frac{1}{n}\right) \forall n \in N$

⇨ $m^*(\phi) \leq m^*\left(-\frac{1}{n}, \frac{1}{n}\right)$

⇨ $m^*(\phi) \leq \frac{2}{n}$

⇨ $0 \leq m^*(\phi) \leq \frac{2}{n}$ for each n. In limiting case $m^*(\phi) = 0$.

1.14 Corollary. If A is countable, $m^* A = 0$

Proof. We know that a countable set is the union of a countable family of singleton. Therefore $A = \cup [x_n]$, which yields

$m^* A = m^* [\cup (x_n)] \leq \Sigma\, m^* [x_n]$ (by the above theorem)

But as already pointed out outer measure of a singleton is zero. Therefore it follows that

$$m^* A \leq 0$$

Since outer measure is always a non – negative real number, $m^* A = 0$.

1.15 Remark: The Sets $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}$ has outer-measure zero.

1.16 Remark: Prove that [0, 1] is uncountable.

Proof: Assume on the contrary that the set [0, 1] is countable, then as we know outer-measure of countable set is zero, then $m^*[0,1] = 0, i.e, l[0,1] = 0$

$i.e., 1 = 0$, a contradiction. therefore [0, 1] is uncountable.

1.17 Corollary. If $m^* A = 0$, then $m^*(A \cup B) = m^*B$.

Proof. Using the above proposition

$m^* (A \cup B) \leq m^*A + m* B$

$= 0 + m^* B$ (i)

Also $B \subset A \cup B$

Therefore $m^* B \leq m^* (A \cup B)$ (ii)

From (i) and (ii) it follows that

$$m^* B = m^* (A \cup B)$$

Note:- Because of the property $m^* (\cup A_n) \leq \Sigma m^* A_n$, the function m* is said to be countably Sub-additive. It would be much better if m* were also countably additive, that is,

if $m^* (\cup A_n) = \sum m^* A_n$.

for every countable collection $[A_n]$ of disjoint sets of real numbers. If we insist on countable additivity, we have to restrict the domain of the function m* to some subset m of the set 2^R of all subsets of R. The members of m are called the measurable subsets of R. That is, to do so we suitably reduce the family of sets on which m* is defined. This is done by using the following definition due to Carathedory.

1.18 Definition. A set E of real numbers is said to be m* measurable, if for every set $A \in R$, we have

$$m^* A = m^* (A \cap E) + m^* (A \cap E^c)$$

Since $A = (A \cap E) \cup (A \cap E^c)$,

It follows from the definition that

$$m^* A = m^* [(A \cap E) \cup (A \cap E^c)] \leq m^* (A \cap E) + m^* (A \cap E^c)$$

Hence, the above definition reduces to:

A set $E \in R$ is measurable if and only if for every set $A \in R$, we have

$m^* A \geq m^* (A \cap E) + m^* (A \cap E^c)$.

For example ϕ is measurable.

1.19 Theorem. Prove that ϕ is measurable set.

Proof: Let A be set of reals, then $m^* A = m^* (A \cap E) + m^* (A \cap E^c)$

Put $E = \phi$

$m^*(A \cap \phi) + m^*(A \cap \phi^c) = m^*(\phi) + m^*(A \cap R)$

$= 0 + m^* A$

$= m^* A$

This implies ϕ is measurable.

1.20 Theorem. Prove that R is measurable set.

Proof: Let A be set of reals, then

$m^* A = m^*(A \cap E) + m^*(A \cap E^c)$

Put E = R

$m^*(A \cap R) + m^*(A \cap R^c) = m^*(A) + m^*(A \cap \phi)$

$= m^*(A) + m^*(\phi)$

$= m^* A + 0$

$= m^* A$

This implies R is measurable.

1.21 Theorem. If $m^* E = 0$, then E is measurable.

Proof. Let A be any set. Then $A \cap E \subset E$ and so

$$m^*(A \cap E) \leq m^* E = 0 \qquad \text{(i)}$$

Also $A \supset A \cap E^c$, and so

$$m^* A \geq m^*(A \cap E^c) = m^*(A \cap E^c) + m^*(A \cap E)$$

as $m^*(A \cap E) = 0$ by (i)

Hence E is measurable.

1.22 Theorem. Every subset of E is measurable if $m^* E = 0$.

Proof: Let F be any subset of E, where $m^* E = 0$.

then since $F \subseteq E$

this implies $m^* F \leq m^* E$

this implies $m^* F \leq 0$

Also $m^* F \geq 0$

therefore $m^* F = 0$.

this implies F is measurable.

1.23 Theorem. Every singleton set is measurable.

Proof: Since outer measure of singleton set is zero and set of measure zero is measurable. Therefore, singleton set is measurable.

1.24 Theorem. Every countable set is measurable.

Proof: Since outer measure of countable set is zero and set of measure zero is measurable. Therefore countable set is measurable.

1.25 Theorem. If a set E is measurable, then so is its complement E^c.

Proof. The definition is symmetrical with respect to E^c, and so if E is measurable, its complement E^c is also measurable.

1.26 Theorem. Union of two measurable sets is measurable.

Proof. Let E_1 and E_2 be two measurable sets and let A be any set. Since E_2 is measurable, we have

$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$ (i)

and since $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup [A \cap E_2 \cap E_1^c]$ (ii)

Therefore by (ii) we have

$m^*[A \cap (E_1 \cup E_2)] \le m^*(A \cap E_1) + m^*[A \cap E_2 \cap E_1^c]$ (iii)

Thus

$m^*[A \cap (E_1 \cup E_2)] + m^*(A \cap E_1^c \cap E_2^c)$

$\le m^*(A \cap E_1) + m^*(A \cap E_2 \cup E_1^c) + m^*(A \cap E_1^c \cap E_2^c)$

$= m^*(A \cap E_1) + m^*(A \cap E_1^c)\ (by\ (i))$

$\le m^* A$ (since E_1 is measurable)

i.e. $m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \le m^* A$

Hence $E_1 \cup E_2$ is measurable.

If E_1 and E_2 are measurable, then $E_1 \cap E_2$ is also measurable.

In fact we note that E_1, E_2 are measurable $\Rightarrow E_1^c, E_2^c$ are measurable $\Rightarrow E_1^c \cup E^c$ is measurable $\Rightarrow (E_1^c \cup E_2^c)^c = E_1 \cap E_2$ is measurable.

Similarly, it can be shown that if E_1 and E_2 are measurable, then $E_1^c \cap E_2^c$ is also measurable.

1.27 Lemma. Difference of two measurable sets is also measurable.

Proof: Let E_1 and E_2 be two measurable sets. Then E_2^c is measurable and hence $E_1 \cap E_2^c = E_1 - E_2$ is measurable, being the intersection of two measurable sets.

1.28 Definition. Algebra or Boolean Algebra: - A collection **A** of subsets of a set X is called an algebra of sets or a Boolean Algebra if

(i) A, B ∈ A => A ∪ B ∈ A
(ii) A ∈ A => A^c ∈ A
(iii) For any two members A and B of A, the intersection A ∩ B is in A.

Because of De Morgan's formulae (i) and (ii) are equivalent to (ii) and(iii).

It follows from the above definition that the collection M of all measurable sets is an algebra. The proof is an immediate consequence of Theorems 1.25 and 1.26.

1.29 Definition. By a Boolean σ - algebra or simply a σ - algebra or Borel field of a collection of sets, we mean a Boolean Algebra A of the collection of the sets such that union of any countable collection of members of this collection is a member of A.

From De Morgan's formula an algebra of sets is a σ - algebra or Borel field if and only if the intersection of any countable collection of members of A is a member of A.

1.30 Lemma. Let A be any set, and $E_1, E_2, \dots, E_n$ a finite sequence of disjoint measurable sets. Then

$$m^*\,(A \cap [U_{i=1}^{n} E_i]) = \Sigma_{i=1}^{n} m^*(A \cap E_i)$$

Proof. We shall prove this lemma by induction on n. The lemma is trivial for

n = 1. Let n > 1 and suppose that the lemma holds for n – 1 measurable sets E_i.

Since E_n is measurable, we have

$m^*\,(X) = m^*\,(X \cap E_n) + m^*\,(X \cap E_n{}^c)$ for every set X ∈ R.

In particular we may take

$X = A \cap [U_{i=1}^{n} E_i]$.

Since $E_1, E_2, \dots, E_n$ are disjoint, we have

$$X \cap E_n = A \cap [U_{i=1}^{n} E_i] \cap E_n = A \cap E_n$$

$$X \cap E_n^c = A \cap [U_{i=1}^{n} E_i] \cap E_n^c = A \cap [U_{i=1}^{n-1} E_i]$$

Hence, we obtain $m^*\,X = m^*(A \cap E_n) + m^*(A \cap [U_{i=1}^{n-1} E_i])$ (i)

But since the lemma holds for $n - 1$ we have

$$m^*(A \cap [\, U_{i=1}^{n-1} E_i]) = \sum_{i=1}^{n-1} m^*(A \cap E_i)$$

Therefore (i) reduces to

$m^*\,X = m * (A \cap E_n) + \Sigma_{i=1}^{n-1} m^*(A \cap E_i)$

$= \sum_{i=1}^{n} m^*(A \cap E_i)$.

Hence the lemma.

1.31 Lemma. Let A be an algebra of subsets and $\{E_i \mid i \in N\}$ a sequence of sets in A. Then there exists a sequence $[D_i \mid i \in N]$ of disjoint members of A such that

$$D_i \subset E_i \;(\, i \in N)$$

$$U_{i \in N} D_i = U_{i \in N} E_i$$

Proof. For every i $\in$ N, let

$$D_n = E_n - (E_1 \cup E_2 \cup \ldots \ldots \cup E_{n-1})$$
$$= (E_n \cap (E_1 \cup E_2 \cup \ldots \ldots \cup E_{n-1}))^c$$
$$= E_n \cap {E_1}^c \cap {E_2}^c \cap \ldots \ldots \cap {E_{n-1}}^c$$

Since the complements and intersections of sets in A are in A, we have each $D_n \in A$. By construction, we obviously have $D_i \subset E_i$ $(i \in N)$

Let D_n and D_m be two such sets, and suppose m < n. Then $D_m \subset E_m$, and so

$D_m \cap D_n \subset E_m \cap D_n$

$= E_m \cap E_n \cap {E_1}^c \cap \ldots .. E_m^c \cap \ldots . \cap E_{n-1}^c$ $\big(using\ (i)\big)$

$= (E_m \cap E_m^c) \cap \ldots . = \phi \cap \ldots \ldots \ldots = \phi$

The relation (i) implies $U_{i \in N} D_i \subset U_{i \in N} E_i$

It remains to prove that

$U_{i \in N} D_i \supset U_{i \in N} E_i$

For this purpose let x be any member of $U_{i \in N} E_i$. Let n denotes the least natural number satisfying x $\in$ E_n. Then we have

$x \in E_n - (E_1 \cup E_2 \cup \ldots \ldots \cup E_{n-1}) = D_n \subset U_{i \in N} D_n$.

This completes the proof.

1.32 Theorem. The collection M of measurable sets is a σ - algebra.

Proof. We have proved already that M is an algebra of sets and so we have only to prove that M is closed with respect to countable union. By the lemma proved above each set E of such countable union must be the union of a sequence $\{D_n\}$ of pairwise disjoint measurable sets. Let A be any set, and let

$E_n = U_{i \in I} D_i \subset E$. Then E_n is measurable and $E_n^c \supset E^c$. Hence

$m^* A = m^*(A \cap E_n) + m^*(A \cap E_n^c) \geq m^*(A \cap E_n) + m^*(A \cap E_n^c)$.

But, by lemma 1.30,

$$m^*(A \cap E_n) = m^*[A \cap (U_{i \in 1} D_i)] = \Sigma_{i=1}^{n} m^*(A \cap D_i)$$

Therefore,

$m^* A \geq \Sigma_{i=1}^{n} m^*(A \cap D_i) + m^*(A \cap E^c)$

Since the left hand side of the inequality is independent of n, we have

$$m^* A \geq \Sigma_{i=1}^{\infty} m^*(A \cap D_i) + m^*(A \cap E^c)$$

$\geq m^*(U_{i \in I}^{\infty} [A \cap D_i]) + m^*(A \cap E^c)$ (by countably subadditivity of m*)

$$= m^*(A \cap U_{i \in I}^{\infty} D_i) + m^*(A \cap E^c)$$

$$= m^*(A \cap E) + m^*(A \cap E_n^c).$$

which implies that E is measurable. Hence the theorem.

1.33 Lemma. The interval (a, ∞) is measurable

Proof. Let A be any set and

$$A_1 = A \cap (a, \infty)$$

$$A_2 = A \cap (a, \infty)^c = A \cap (-\infty, a].$$

Then we must show that

$$m^* A_1 + m^* A_2 \leq m^* A.$$

$If\ m^* A = \infty$, then there is nothing to prove. If m* A $< \infty$, then given ∈ > 0 there is a countable collection $\{I_n\}$ of open intervals which cover A and for which

$$\Sigma l(I_n) \leq m^* A + \in$$

Let $I_n' = I_n \cap (a, \infty)\ and\ I_n'' = I_n \cap (-\infty, a)$. Then I_n' and I_n'' are intervals (or empty) and $l(I_n) = l\,(I_n') + l(I_n'') = m^*(\,I_n') + m^*(\,I_n'')$

Since $A_1 \subset U I_n'$, we have

$m^* A_1 \leq m^*(U I_n') \leq \Sigma\, m^* I_n'$, (iii)

and since, $A_2 \subset U\, I_n''$, we have

$m^* A_2 \leq m^*(U\, I_n'') \leq \Sigma\, m^* I_n''$, (iv)

Adding (iii) and (iv) we have

$m^* A_1 + m^* A_2 \leq \Sigma\, m^* I_n' + \leq \Sigma\, m^* I_n''$

$= \Sigma\,(m^* I_n' + \leq m^* I_n'')$

$= \Sigma\, l\,(I_n) \qquad [by\ (ii)]$

$\leq m^* A + \in \quad [by\ (i)]$

But ∈ was arbitrary positive number and so we must have $m^* A_1 + m^* A_2 \leq m^* A$.

1.34 Definition. The collection ß of Borel sets is the smallest σ - algebra which contains all of the open sets.

1.35 Theorem. Every Borel set is measurable. In particular each open set and each closed set is measurable.

Proof. We have already proved that (a, ∞) is measurable. So we have

$(a, \infty)^c = (-\infty, a]$ measurable.

Since $(-\infty, b) = U_{n=1}^{\infty} \left((-\infty, b - \frac{1}{n}]\right)$ and we know that countable union of measurable sets is measurable, therefore (- ∞, b) is also measurable. Hence each open interval,

$(a, b) = (-\infty, b) \cap (a, \infty)$ is measurable, being the intersection of two measurable sets. But each open set is the union of countable number of open intervals and so must be measurable (The measurability of closed set follows because complement of each measurable set is measurable).

Let M denote the collection of measurable sets and C the collection of open sets. Then

C ⊂ M. Hence ß is also a subset of M since it is the smallest σ - algebra containing C . So each element of ß is measurable. Hence each Borel set is measurable.

1.36 Definition. If E is a measurable set, then the outer measure of E is called the Lebesgue Measure of E, is denoted by m. Thus, m is the set function obtained by restricting the set function m* to the family M of measurable sets. Two important properties of Lebesgue measure are summarized by the following theorem.

1.37 Theorem. Let $\{E_n\}$ be a sequence of measurable sets. Then

$m(\cup E_i) \leq \Sigma\, m\, E_i$

If the sets E_n are pairwise disjoint, then

$$m(\cup E_i) = \Sigma\, mE_i\,.$$

Proof. The inequality is simply a restatement of the sub-additivity of m*. If $\{E_i\}$is a finite sequence of disjoint measurable sets. So we apply lemma 1.30 replacing A by R. That is , we have

$$m^*(R \cap [U_i^n E_i]) = \Sigma_{i=1}^n\, m^*\,(R \cap E_i)$$

$$m^*(U_i^n E_i) = \sum_i^n m^* E_i$$

and so m is finitely additive..

Let $\{E_i\}$ be an infinite sequence of pairwise disjoint sequence of measurable sets. Then

And so $\quad U_{i=1}^\infty\, E_i \supset U_{I=1}^n E_n$

$m(U_{i=1}^\infty\, E_i) \geq m(U_{(i=1)}^\infty\, E_i) = \Sigma_{i=1}^\infty\, m\, E_i$

Since the left-hand side of this inequality is independent of n, we have

$m(U_{i=1}^\infty\, E_i) \geq \Sigma_{i=1}^\infty\, m\, E_i$

The reverse inequality follows from countable sub-additivity and we have

$m(U_{i=1}^\infty\, E_i) = \Sigma_{i=1}^\infty\, m\, E_i$

Hence the theorem is proved.

1.38 Theorem. Let $\{E_n\}$ be an infinite sequence of measurable sets such that $E_{n+1} \subset E_n$ for each n. Let $mE_1 < \infty$. Then

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} mE_n$$

Proof. Let $E = \bigcap_{i=1}^{\infty} E_i$ and let $F_i = E_i - E_{i-1}$. Then since $\{E_n\}$ is a decreasing sequence. We have $\bigcap F_i = \phi$.

Also we know that if A and B are measurable sets then their difference $A - B = A \cap B^c$ is also measurable. Therefore each F_i is measurable. Thus $\{F_i\}$ is a sequence of measurable pairwise disjoint sets.

$$\text{Now} \qquad \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (E_i - E_{i+1})$$

$$= \bigcup_{i=1}^{\infty} (E_i \cap E_{i+1}^c)$$

$$= E_1 \cap (\cup \ E_i^c)$$

$$= E_1 \cap \left(\bigcap_{i=1}^{\infty} E_i \right)^c$$

$$= E_1 \cap E^c$$

$$= E_1 - E$$

Hence

$$m\left(\bigcup_{i=1}^{\infty} F_i \right) = m(E_1 - E)$$

$$\Rightarrow \sum_{i=1}^{\infty} mF_i = m(E_1 - E)$$

$$\Rightarrow \sum_{i=1}^{\infty} m(E_i - E_{i+1}) = m(E_1 - E) \qquad \ldots \quad \text{(i)}$$

Since $E_1 = (E_1 - E) \cup E$, therefore

$$mE_1 = m(E_1 - E) + m(E)$$

$$\Rightarrow mE_1 - mE = m(E_1 - E) \quad (\text{since } mE \leq mE_1 < \infty) \ \ldots \text{(ii)}$$

Again

$$E_i = (E_i - E_{i+1}) \cup E_{i+1}$$

$$\Rightarrow mE_i = m(E_i - E_{i+1}) + mE_{i+1}$$

$$\Rightarrow mE_i - mE_{i+1} = m(E_i - E_{i+1}) \ (\text{ since } E_{i+1} \subset E_i \ ) \ \ldots \text{(iii)}$$

Therefore (i) reduces to

$$mE_1 - mE = \sum_{i=1}^{\infty} (mE_i - mE_{i+1}) \ \ (\text{ using (ii)and (iii)})$$

$$= \lim_{n\to\infty} \sum_{i=1}^{\infty} (mE_i - mE_{i+1})$$

$$= \lim_{n\to\infty} [mE_1 - mE_2 + mE_2 - mE_3 \ldots - mE_{n+1}]$$

$$= \lim_{n\to\infty} [mE_1 - mE_{n+1}]$$

$$= mE_1 - \lim_{n\to\infty} E_{n+1}$$

$$\Rightarrow mE = \lim_{n\to\infty} mE_n$$

$$\Rightarrow m(\cap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} mE_n$$

1.39 Remark. Show that the condition $m(E_1) < \infty$ is necessary in the above theorems.

Solution. Let $E_n = [n, \infty)$

Then, $E_1 = [1, \infty)$

$\Rightarrow m(E_1) = m[1, \infty) = \infty$

We show that the proposition of decreasing sequence does not hold in this case i.e. we want to show that

$m(\cap_{n=1}^{\infty} E_n) \neq \lim_{n\to\infty} (E_n)$

Clearly, $E_{n+1} \subset E_n$ for all n

Now, $E_n = [n, \infty) \supset [n, 2n]$

$\Rightarrow$ $m(E_n) \geq m[n, 2n] = n$

$\Rightarrow$ $m(E_n) \geq n$

$\Rightarrow \lim_{n\to\infty} m(E_n) = \infty$ … (1)

Now, we claim that $m(\cap_{n=1}^{\infty} E_n) = m(\cap_{n=1}^{\infty} [n, \infty)) = 0$

For if, $\cap_{n=1}^{\infty} E_n \neq \phi \Rightarrow$ there exists $x \in \cap_{n=1}^{\infty} E_n$

$\Rightarrow$ $x \in [n, \infty)$ for all $n \in N$

Let $x \in R$, so by Archmedian property, we can find a positive integer n_0 such that

$$n_0 \leq x < n_0 + 1$$

$\Rightarrow$ $x \notin [n_0 + 1, \infty)$, a contradiction

$\therefore \cap_{n=1}^{\infty} E_n = \phi$

$\Rightarrow$ $m(\cap_{n=1}^{\infty} E_n) = 0$ … (2)

From (1) and (2), we have

$$m(\cap_{n=1}^{\infty} E_n) \neq \lim_{n\to\infty} (E_n)$$

S0, theorem does not hold in this case.

1.40 Theorem. Let $\{E_n\}$ be an increasing sequence of measurable sets. i.e. a sequence with $E_n \subset E_{n+1}$ for each n. Let mE_1 be finite, then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n \, .$$

Proof. The sets E_1, $E_2 - E_1$, $E_3 - E_2$, …, $E_n - E_{n+1}$ are measurable and are pairwise disjoint . Hence

$$E_1 \cup (E_2 - E_1) \cup \ldots \cup (E_n - E_{n-1}) \cup \ldots$$

is measurable and

$$m[E_1 \cup (E_2 - E_1) \cup \ldots \cup (E_n - E_{n-1}) \cup \ldots]$$

$$= mE_1 + \sum_{i=2}^{n} m\,(E_i - E_{i-1})$$

$$= mE_1 + \lim_{n\to\infty} \sum_{i=2}^{n} m(E_i - E_{i-1})$$

But

$E_1 \cup (E_2 - E_1) \cup \ldots \cup (E_n - E_{n-1}) \cup \ldots$ is precisely $\bigcup_{i=1}^{\infty} E_n$

Moreover,

$$\sum_{i=2}^{n} m\,(E_i - E_{i-1}) = \sum_{i=2}^{n} (mE_i - mE_{i-1})$$

$$= (mE_2 - mE_1) + (mE_3 - mE_2) + \cdots + (mE_n - mE_{n-1})$$

$$= mE_n - mE_1$$

Thus we have

$$m\left[\bigcup_{i=1}^{\infty} E_i\right] = mE_1 + \lim_{n\to\infty}[mE_n - mE_1]$$

$$= \lim_{n\to\infty} mE_n$$

1.41 Definition : The symmetric difference of the sets A and B is the union of the sets A-B and B-A . It is denoted by $\Delta\, B$.

1.42 Theorem. If $m(E_1 \,\Delta\, E_2) = 0$ and E_1 is measurable, then E_2 is measurable. Moreover $mE_2 = mE_1$.

Proof . We have

$E_2 = [E_1 \cup (E_2 - E_1)] - (E_1 - E_2)$ …(i)

By hypothesis, both $E_2 - E_1$ and $E_1 - E_2$ are measurable and have measure zero. Since E_1 and E_2-E_1 are disjoint, $E_1 \cup (E_2 - E_1)$ is measurable and

$m[E_1 \cup (E_2 - E_1)] = mE_1 + 0 = mE_1$. But, since

$E_1 - E_2 \subset [E_1 \cup (E_2 - E_1)]$,

it follows from (i) that E_2 is measurable and

$$\begin{aligned} mE_2 &= m[\,E_1 \cup (E_1 - E_2)] - m(E_1 - E_2) \\ &= mE_1 - 0 = mE_1\,. \end{aligned}$$

This completes the proof.

1.43 Definition. Let x and y be real numbers in [0,1]. The **sum modulo 1 of x and y** , denoted by $x \overset{0}{+} y$, is defined by

$$x \overset{0}{+} y = \begin{cases} x + y \; if \; x + y < 1 \\ x + y - 1 \; if \; x + y \geq 1 \end{cases}$$

It can be seen that $\overset{0}{+}$ is a commutative and associative operation which takespair of numbersin [0,1) into numbers in [0,1).

If we assign to each $x \in [0,1)$ the angle $2\pi x$ then addition modulo 1 corresponds to the addition of angles.

If E is a subset of [0,1), we define the translation modulo 1 of E to be the set

$E \overset{0}{+} y$ = [z $|z =$ x $\overset{0}{+}$ y for some x $\in E$].

If we consider addition modulo 1 as addition of angles, translation module 1 by y corresponds to rotation through an angle of $2\pi y$.

We shall now show that Lebesgue measure is invariant under translation modulo 1.

1.44 Definition. Let x and y be real numbers in [0,1). The **sum modulo 1 of x and y**, denoted by $\overset{0}{+}\, y$, is defined by

$$x \overset{0}{+} y = \begin{cases} x + y \; if \; x + y < 1 \\ x + y - 1 \; if \; x + y \geq 1 \end{cases}$$

Clearly $x \overset{0}{+} y \in [0,1)$

It can be seen that $\overset{0}{+}$ is a commutative and associative operation which takes pair of numbers in [0,1) into numbers in [0,1).

1.45 Definition. If E is a subset of [0,1), we define the translation modulo 1 of E to be the set

$E \overset{0}{+} y$ = [z | $z =$ x $\overset{0}{+}$ y for some x ∈ E].

We shall now show that Lebesgue measure is invariant under translation modulo 1.

1.46 Lemma. Let $E \subset [0,1)$ be a measurable set. Then for each $y \in [0,1)$ the set $E \overset{0}{+} y$ is measurable and m $(E \overset{0}{+} y) = mE.$

Proof. Let E_1= E$\cap [0, 1 - y)$ and E_2 = E $\cap [1 - y, 1)$. Then E_1 and E_2 are disjoint measurable sets whose union is E, and so, mE = mE_1 + mE_2.

we observe that

$E_1 \overset{0}{+} y =$\{ $x \overset{0}{+} y$: x ∈ E_1\}

$$= \begin{cases} x + y \; if \; x + y < 1 \\ x + y - 1 \; if \; x + y \geq 1. \end{cases} \qquad x \in E_1$$

But for x ∈ E_1, we have x + y < 1 and so

$E_1 \overset{0}{+} y = \{ x + y, x \in E_1\} = E_1 + y.$

and hence $E_1 \overset{0}{+} y$ is measurable. Thus

m $(E_1 \overset{0}{+} y) = m(E_1 + y) = m(E_1),$

since m is translation invariant. Also $E_2 \overset{0}{+} y = E_2 + (y - 1)$ and so $E_2 \overset{0}{+} y$ is measurable and m$(E_2 \overset{0}{+} y) = mE_2.$ But

$$E \overset{0}{+} y = (E_1 \overset{0}{+} y) \cup (E_2 \overset{0}{+} y)$$

And the sets $(E_1 \overset{0}{+} y)$ and $(E_2 \overset{0}{+} y)$ are disjoint measurable sets. Hence $E \overset{0}{+} y$ is measurable and

$$m\,(E \overset{0}{+} y) = m[(E_1 \overset{0}{+} y) \cup (E_2 \overset{0}{+} y)]$$

$$= m(E_1 \overset{0}{+} y) + m(E_2 \overset{0}{+} y)$$

$$= m(E_1) + m(E_2)$$

$$= m(E).$$

This completes the proof of the lemma.

1.47 Theorem: Prove that there exists a non-measurable set in interval [0,1).

Proof: First we define an equivalence relation in the set I= [0,1), By saying that x and y are equivalent i.e., $x \sim y$ if and only if x-y is a rational number.

If x-y is a rational number, we say that x and y are equivalent and write x-y. It is clear that $x \sim x\,;x \sim y \Rightarrow y \sim x$ and $x \sim y\,, y \sim z \;\Rightarrow x \sim z$. Thus '$\sim$' is an equivalence relation in I.

Hence the relation ~ partitions the set I = [0,1) into mutually disjoint equivalence classes, that is, classes such that any two elements of one class differ by a rational number, while any two elements of different classes differ by an irrational number.

Construct a set P by choosing exactly one element from each equivalence classes. Now we claim that P is a non-measurable set.

Let $< r_i >\,_{i=0}^{\infty}$ be a sequence of the rational numbers in [0,1) with $r_0 = 0$ and define $P_i = P \overset{0}{+} r_i$.

(translation modulo 1 of P)

Then $P_0 = P$.

We further prove that (i) $P_i \cap P_j = \emptyset, i \neq j$.

(ii) $\bigcup_n P_n = [0,1)$

Proof: (i) Let $P_i \cap P_j \neq \emptyset, i \neq j$.

Let $x \in P_i \cap P_j$. => $x \in P_i and\ x \in P_j$

Then$\exists\ p_i, p_j \in P\ such\ that\ x = p_i \overset{0}{+} r_i$

$$x = p_j \overset{0}{+} r_j$$

- ⇨ $p_i \overset{0}{+} r_i = p_j \overset{0}{+} r_j$
- ⇨ p_i-$p_j = r_j - r_i$ *is a rational number.*
- ⇨ $p_i \sim p_j$ *is a rational number.*

  i.e., $p_i \sim p_j$

  => p_i *and* p_j *are in same equivalence class.*

But P has only one element from each equivalence class, therefore we must have $p_i = p_j$ *i.e.*, $i = j$

But $\neq j$. Hence a contradiction.

Hence $P_i \cap P_j \neq \emptyset, i \neq j$.

that is, < P_i >is a pair wise disjoint sequence of sets.

(ii) Clearly each $P_i \subset [0, 1)$

$\bigcup_i P_i \subset [0, 1)$. Let x be any element of [0, 1) = I.

But I is partitioned into equivalent classes therefore x lies in one of the equivalence classes.

- ⇨ x is equivalent to an element say y of P.
- ⇨ x-y is a rational number say r_i.
- ⇨ x-y = r_i
- ⇨ x = y + r_i

  $= y \overset{0}{+} r_i$.

  $x \in P \overset{0}{+} r_i$

- ⇨ x$\in P_i$
- ⇨ x is in some P_i.

$$Therefore\ [0,1) \subseteq \bigcup_i P_i$$

$$therefore, [0,1) = \bigcup_i P_i.$$

Now we prove P is non-measurable.

Assume that P is measurable, then clearly each P_i is measurable.

And $m(P_i) = m\left(P \overset{0}{+} r_i\right)$

$= m(P)$ for each i.

Therefore, $m(\bigcup_i P_i) = \sum_i m(P_i) = \sum_{i=0}^{\infty}(P)$

$$= \begin{cases} 0 & if\ m(P) = 0 \\ \infty & if\ m(P) > 0 \end{cases}$$

But

$$m\left(\bigcup_i P_i\right) = m[(0,1)] = l(0,1) = 1, contadiction.$$
$$Therefore\ P\ is\ non-measurable\ set.$$

1.48 Example. The cantor set is uncountable with outer measure zero.

Solution. We already know that cantor set is uncountable. Let C_n denote the union of the closed intervals left at the nth stage of the construction. We note that C_n consists of 2^n closed intervals, each length 3^{-n}. Therefore

$$m^*C_n \leq 2^n.3^{-n} \qquad (\therefore\ m^*C_n = m^*(\cup F_n) = \sum m^*F_n)$$

But any point of the cantor set C must be in one of the intervals comprising the union C_n, for each n $\in N$, and as such $C \subset C_n$ for all n $\in N$. Hence

$$m^*C \leq m^*C_n \leq \left(\frac{2}{3}\right)^n$$

This being true for each n $\in N$, letting $n \to \infty$ gives $m^*C = 0$.

1.49 Example. If E_1 and E_2 are any measurable sets, show that

$$M(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Proof. Let A be any set. Since E_1 is measurable,

$$m^*A = m^*(A \cap E_1) + m^*(A \cap {E_1}^c).$$

We set $A = E_1 \cup E_2$ and we have

$$m^*(E_1 \cup E_2) = m^*[(E_1 \cup E_2) \cap E_1] + m^*[(E_1 \cup E_2) \cap {E_1}^c]$$

Adding $m(E_1 \cup E_2)$ to both sides we have

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = mE_1 + m^*[(E_1 \cup E_2) \cap {E_1}^c] + m(E_1 \cap E_2) \ \dots (1)$$

But

$$E_2 = [(E_1 \cup E_2) \cap {E_1}^c] \cup (E_1 \cup E_2).$$

Therefore

$$m\{[(E_1 \cup E_2) \cap {E_1}^c] \cup (E_1 \cup E_2)\} = mE_2$$

Hence (1) reduces to

$$M(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

1.50 Theorem. Let E be any set. Then given $\in > 0$, there is an open set $O \supset E$ such that $m^*O < m^*E + \in$.

Proof. There exists a countable collection [I_n] of open intervals such that

$E \subset \bigcup_n I_n$ and

$$\sum_{n=1}^{\infty} l(I_n) < m^* E + \in .$$

$$put\ O = \bigcup_{n=1}^{\infty} I_n.$$

Then O is an open set and

$$m^*O = m^*\left(\bigcup_{n=1}^{\infty} I_n\right)$$

$$\leq \sum_{n=1}^{\infty} m^*I_n$$

$$= \sum_{n=1}^{\infty} l(I_n) < m^*E + \in .$$

1.51 Theorem. Let E be a measurable set. Given $\in > 0$, there is an open set

$O \supset E$ such that $m^*(O \setminus E) < \in$.

Proof. Suppose first that m E < ∞. Then by the above theorem there is an open set $O \supset E$ such that

$$m^*O < m^*E + \in$$

Since the sets O and E are measurable, we have

$$m^*(O \setminus E) = m^*O - m^*E < \in.$$

Consider now the case when m E = ∞ . Write the set **R** of real number as a union of disjoint finite intervals; that is,

$$\mathbf{R} = \bigcup_{n=1}^{\infty} \mathbf{I_n}.$$

Then, if $E_n = E \cap I_n$, $m(E_n) < \infty$. We can, thus, find open sets $O_n \supset E_n$ such that

$$m^*(O_n - E_n) < \frac{\in}{2^n}.$$

Define $O = \bigcup_{n=1}^{\infty} O_n$. Clearly O is an open set such that $O \supset E$ and satisfies

$$O - E = \bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} (O_n - E_n)$$

$$m^*(O - E) \leq \sum_{n=1}^{\infty} m^*\left(\frac{O_n}{E_n}\right) < \in .$$

1.52 F_σ and G_δ Sets:

A set which is countable(finite or infinite) union of closed sets is called an F_σ sets. Note: The class of all $\boldsymbol{F_\sigma}$ sets is denoted by F_σ. This **F** stands for ferme(closed) and σ for summe(sum).

Example: 1. A closed set.

2. A countable set

3. A countable union of F_σ set.

4. An open interval (a, b) since

$(a, b) = U_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$and hence an open set.

***G_δ*- set:**

A set which is countable intersection of open sets is a G_δ set.

Note: The class of all G_δ sets is denoted by This G stands for region and $\delta$for intersection. The complement of $\boldsymbol{F_\sigma}$ set is a $\boldsymbol{G_\delta}$ set and conversely.

Example: 1. An open set in particular an open interval.

2. A closed set

3. A countable intersection of G_δ set.

4. A closed interval [a, b] since

$[a, b] = \cap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n}).$

1.53 Theorem. Let E be any set then

(a) Given ε > 0,∃an open set $O \supset E$such that $m^*(O) < m^*(E) + \varepsilon$

$(b) \exists a\ G_\delta\ set\ G \supset E$such that m*(E) = m*(G).

Proof: (a) By definition, m* $(E) = inf\ \sum_n l\ (I_n)$, where $E \subseteq \bigcup_n I_n$

if m*(E) = ∞, then clearly result is true. If m*(E) < 0, there is a countable collection {In} of open intervals such that

$E \subseteq U_n I_n$and $m^*(E) + \varepsilon > \sum_n l\ (I_n)$ (1)

Let $O = E \subseteq U_n I_n$, then O is an open set and $O \supset E$

Also $m^*(O) = m^*(E \subseteq U_n I_n)$

$$\leq \sum_n m^*(I_n)$$

m*(O) < m*(E) + ε [from(1)]

(b) Take ε $=\frac{1}{n}\ \forall n \in N$ Then by above part, for each $n \in N$,∃an open set$O_n \supset E$such that

$$m^*(O_n) < m^*(E) + \frac{1}{n}$$

Now define $G = \cup_{n=1}^{\infty} O_n$, then G is a G_δ set.

Also, since each $O_n \supset$ E

therefore $\cup_{n=1}^{\infty} O_n \supset E$ this implies $G \supset E$

≥ m*(E) ≤ m*(G) (2)

Also $G = \cup_{n=1}^{\infty} O_n \subseteq O_n\ \forall n$

$m^*(G) \leq m^*(O_n)$ for each n

$< m^*(E) + \frac{1}{n}$, for each n

in limiting case, we have

m*(G) ≤ m*(E) (3)

Then from (2) and (3), we have

m*(G) = m*(E).

1.54 Theorem. Let E be any set,then the following five statements are equivalent.

(i) E is measurable.

(ii) For given **ε** > 0 , ∃an open set $O \supset E$ such that **m*(O – E) < ε**

(iii) There exist a set G in G_δ with $E \subset G$, m*(G – E) = 0

(iv) For given ε > 0 , ∃ an closed set $F \subset E$ such that **m*(E – F) < ε**

(v) There exist a set F in F_σ with $F \subset E$, m*(E – F) = 0

Proof. Ist we prove (i) implies (ii)

Let E be a measurable set.

Now two cases arrive

Case (i) m*(E) < ∞.

By definition, for given **ε**> 0,there is a countable collection {I_n} of open intervals such that

$E \subseteq \bigcup_n I_n$ and m*(E) + ε > $\sum_n l\,(I_n)$.......(1)

Let O = $E \subseteq \bigcup_n I_n$, then O is an open set and $O \supset E$

Also m*(O) =m*($E \subseteq \bigcup_n I_n$)

m*(O) $\leq \sum_n m*(I_n)$

m*(O) < m*(E) + ε [from(1)]

m*(O)- m*(E) < ε

O = $(O - E) \cup E$

m*(O) = m*($(O - E) \cup E$) = m*(O-E)+m*(E)

m*(O-E) = m*(O) - m*(E)

$\Longrightarrow$ m*(O-E) < ε

Case (ii) If m*(E) = ∞

We know that set of real number can be written as countable union of disjoint open intervals

$$R = \bigcup_{n=1}^{\infty} I_n$$

Then $E = E \cap R$

$$= E \bigcap \bigcup_{n=1}^{\infty} I_n$$

$$= \bigcup_{n=1}^{\infty} (E \cap I_n)$$

$$\Longrightarrow E = \bigcup_{n=1}^{\infty} E_n \text{ , } where\ E_n = E \cap I_n$$

clearly each E_n is measurable and m(E_n) is finite.

Because $E_n = E \cap I_n \subseteq I_n$

m*(E_n) ≤ l(I_n) <∞

Then by case (i), for each $n \in N$, ∃ an open set $O_n \supset E_n$ such that

m*(O_n - E_n) <$\frac{\varepsilon}{2^n}$

Let us define $O = \bigcup_{n=1}^{\infty} O_n$

Then O is an open set containing E

Now (O-E) = $\bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} (O_n - E_n)$

m*(O-E) ≤ m*($\bigcup_{n=1}^{\infty} (O_n - E_n)$)

$\leq \sum_{n=1}^{\infty} m^* (O_n - E_n)$

$\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$

$= \varepsilon$

$\Longrightarrow$ m*(O-E) < ε

Now (ii) $\Longrightarrow$ (iii)

Let (ii) holds, then for each $n \in N$, $\exists$an open set $O_n \supset E$such that

m*(O_n-E) <$\frac{1}{n}$

Let us define = $\bigcap_{n=1}^{\infty} O_n$, then G is a $G_\delta$set.

Also since each $O_n \supset E$

therefore $\bigcap_{n=1}^{\infty} O_n \supset E$ this implies $G \supset E$

G-E = $\bigcap_{n=1}^{\infty} O_n - E \subseteq O_n - E$

m*(G-E) ≤ m*(O_n-E) < $\frac{1}{n}$

Since n is arbitrary

m*(G-E) $\leq$ 0

$\Longrightarrow$ m*(G-E) = 0.

Now (iii) $\Longrightarrow$ (i)

Let (iii) holds, then for given set E, $\exists$ a G_δ set $G \supset E$ such that m*(G-E) = 0

$\Longrightarrow$ G - E is measurable.

Now E = G- (G-E)

Now E is measurable being difference of two measurable sets.

Thus (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii)

Now to show (i) $\Longrightarrow$ (iv)

Let (i) holds, and ε > 0 be given

then by (ii), for given set E^c, $\exists$ an open set $G \supset E^c$ such that m*(G - E^c) < ε

Since $G \supset E^c \Longrightarrow G^c \subseteq E$

Let F = G^c

then F is a closed set contained in E,

Now E-F = $E \cap F^c$ = E∩G = G∩E = G – E^c

Now m*(E-F) = m*(G-E^c) < ε

m*(E-F) < ε.

To Show (iv) $\Longrightarrow$ (v)

Let (iv) holds, then for each $n \in N$,$\exists$a closed set $F_n \subset E$ such that

$m^*(E - F_n) < \frac{1}{n}$

Let us define $F = \bigcup_{n=1}^{\infty} F_n$

Then F is a F_σ set.

Also, since each $F_n \subset E \implies \bigcup_{n=1}^{\infty} F_n \implies F \subseteq E$

Now E – F = E - $\bigcup_{n=1}^{\infty} F_n \subseteq E - F_n$

$\implies$ m*(E-F) $\leq$ **$m^*(E - F_n) < \frac{1}{n}$**

$\implies$ m*(E-F) $\leq \frac{1}{n}$

Since n is arbitrary.

m*(E-F) $\leq$ 0

$\implies$ m*(E-F) = 0.

Now (v) $\implies$ (i)

Let (v) holds, then for general E, ∃ a F_σ set F such that m*(E-F) = 0

$\implies$ E-F is measurable.

$$E = (E - F) \cup F$$

$\implies$ E is measurable.

This completes the proof.

(b) Take $\varepsilon = \frac{1}{n}$ $\forall n \in N$

Then by above part, for each $n \in N$, ∃ an open set $O_n \supset E$ such that

$m^*(O_n) < m^*(E) + \frac{1}{n}$

Now define $G = \bigcap_{n=1}^{\infty} O_n$, then G is a $G_{\delta}-$set.

Also since each $O_n \supset E$

therefore$\bigcap_{n=1}^{\infty} O_n \supset E$

this implies $G \supset E$

$\implies$ m*(E) $\leq$ m*(G) ......(2)

Also G = $\bigcap_{n=1}^{\infty} O_n \subseteq O_n \forall n$

$m^*(G) \leq m^*(O_n)$ for each n < m*(E) +$\frac{1}{n}$, for each n in limiting case, we have

m*(G) $\leq$ m*(E)...(3)

Then from (2) and (3), we have

m*(G) = m*(E).

1.55 Theorem. Let E be a set with m* E < ∞ . Then E is measurable iff given ∈ > 0 , there is a finite union B of open intervals such that m*(E Δ B) < ∈.

Proof. Suppose E is measurable and let ∈ > 0 be given. The (as already shown) there exists an open set O ⊃ E such that m* (O- E) < $\frac{\in}{2}$. As m*E is finite, so is m*O. Since the open set O can be written as the union of countable (disjoint) open intervals {Ii}, there exists an n ∈ **N** such that

$$\sum_{i=n+1}^{\infty} l(I_i) < \frac{\in}{2} \text{ (In fact m* O = } = \sum_{i=n+1}^{\infty} l(I_i) < \infty \Rightarrow \sum_{i=n+1}^{\infty} l(I_i) < \frac{\in}{2} \text{ because m* O} < \infty \text{)}$$

Set B = $\bigcup_{i=1}^{n} I_i$. . Then E ΔB = (E – B) ∪ (B -E) ⊂ (O - B) ∪ (O -E) . Hence

$$m^*(E \,\Delta\, B) \le m^* \left(\bigcup_{i=1}^{n} I_i\right) + m^*(O\text{-}E) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Conversely, assume that for a given ∈ > 0 , there exists a finite union B = $\bigcup_{i=1}^{n} I_i$. if open intervals with m* (E Δ B) < ∈ . Then using "Let ∈ be any set. The given ∈ > 0 there exists an open set O ⊃ E such that m* O < m* E + ∈ there is an open set O ⊃ E such that

$$m^* O < m^* E + \in \qquad \text{(i)}$$

If we can show that m* (O — E) is arbitrary small, then the result will follow from "Let E be

set. Then the following are equivalent (i) E is measurable and (ii) given ∈ > 0 there is an open set O ⊃ E such that m * (O — E) < ∈". Write S =$\bigcup_{i=1}^{n}(I_i \cap O)$. Then S ⊂ B and so

S Δ E = (E - S) ∪ (S - E) ⊂ (E — S) ∪ (B — E) . However,

E \ S = (E ∩ O^C) ∪ (E ∩ B^C) = E — B, because E ⊂ O . Therefore

S Δ E ⊂ (E — B) ∪ (B — E) = E Δ B , and as such m* (S Δ E) < ∈ . However,

E ⊂ S ∪ (S Δ E)

and so m* E < m* S + m* (S Δ E)

$$< m^*S + \in \qquad \text{(ii)}$$

Also,

$$O - E = (O - S) \cup (S \,\Delta\, E)$$

Therefore

$$\begin{aligned} m^* (O \setminus E) &< m^* O - m^* S + \in \\ &< m^* E + \in - m^* S + \in && \text{(using(i))} \\ &< m^* S + \in + \in - m^*S + \in && \text{(using(ii))} \\ &< m^*S + \in + \in - m^* S + \in \\ &= 3 \in . \end{aligned}$$

Hence E is measurable.

SECTION – II

MEASURABLE FUNCTIONS

Measurable Function: An extended real valued function **f** defined on a measurable set E is said to be measurable function if $\{x|\ f(x) > \alpha\}$ is measurable for each real number α.

2.1 Theorem. A constant function with a measurable domain is measurable.

Proof: Let **f** be a constant function with a measurable domain E and Let **f** : E→ R be a constant function i.e., **f**(x) = k $\forall\ x \in E$ *and* k *is constant*.

To show that $\{x|\ f(x) > \alpha\}$ is measurable for each real number α.

$$\{x|\ \mathbf{f}(x) > \alpha\} = \begin{cases} E, & k > \alpha \\ \varphi, & k = \alpha \\ \varphi, & k < \alpha \end{cases}$$

Since both φ *and* E are measurable, it follows that the set $\{x|\ \mathbf{f}(x) > \alpha\}$ and hence **f** is measurable.

2.2 Theorem. Let f be an extended real valued function defined on a measurable set E, Then f is said to be measurable (Lebesgue function) if for any real α any one of the following four conditions is satisfied.

(a) $\{x|\ f(x) > \alpha\}$ is measurable

(b) $\{x|\ f(x) \geq \alpha\}$ is measurable

(c) $\{x|\ f(x) < \alpha\}$ is measurable

(d) $\{x|\ f(x) \leq \alpha\}$ is measurable.

Proof: We show that these four conditions are equivalent. First of all we show that (a) and (b) are equivalent. Since

$$\{x|\ f(x) > \alpha\} = \{x|\ f(x) \leq \alpha\}^c$$

And also we know that complement of a measurable set is measurable, therefore (a) ⟹(d) and conversely.

Similarly since (b) and (c) are complement of each other, (c) is measurable if (b) is measurable and conversely.

Therefore, it is sufficient to prove that (a) ⟹(b) and conversely.

Firstly we show that (b) ⟹ (a).

The set $\{x|\ f(x) \geq \alpha\}$ is given to be measurable.

Now

$$\{x|\ f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x|\ f(x) \geq \alpha + \frac{1}{n}\}$$

But by (b), $\{x|\ f(x) \geq \alpha + \frac{1}{n}\}$ is measurable. Also we know that countable union of measurable sets is measurable. Hence $\{x|\ f(x) > \alpha\}$ is measurable which implies that (b) ⟹(a).

Conversely, let (a) holds. We have

$$\{x|\ f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x|\ f(x) \geq \alpha - \frac{1}{n}\}$$

The set $\{x|\ f(x) > \alpha - \frac{1}{n}\}$ is measurable by (a). Moreover, intersection of measurable sets is also measurable . Hence $\{x|\ f(x) \geq \alpha\}$ is also measurable . Thus (a) ⟹ (b).

Hence the four conditions are equivalent.

2.3 Remark: We can say that f is measurable function if for any real number α, any of the four conditions in the above theorem holds.

2.4 Lemma. If α is an extended real number then these four conditions imply that $\{x|\ f(x) = \alpha\}$ is also measurable.

Proof. Let α be a real number, then

$$\{x|\ f(x) = \alpha\} = \{x|\ f(x) \geq \alpha\} \cap \{x|\ f(x) \leq \alpha\}.$$

Since $\{x|\ f(x) \geq \alpha\}$ and $\{x|\ f(x) \leq \alpha\}$ are measurable by conditions (b) and (d), the set $\{x|\ f(x) = \alpha\}$ is measurable being the intersection of measurable sets.

Suppose $\alpha = \infty$. Then $\{x|\ f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x|\ f(x) > n\}$

Which is measurable by the condition (a) and the fact intersection of measurable sets is measurable.

Similarly when $= -\infty$, then

$\{x|\ f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x|\ f(x) < -n\}$, which is again measurable by conditions (c).Hence the results follows.

2.5 Theorem: If f is measurable function on each of the sets in a countable collection $\{E_i\}$ of disjoint measurable sets, then f is measurable on E = $\bigcup_i E_i$.

Proof: Let E = $\bigcup_i E_i$. Then E is measurable being countable union of measurable sets is measurable.

Let α be any real number.

Consider the set $\{x \in E|\ f(x) > \alpha\} = \bigcup_i \{x \in E_i: f(x) > \alpha\}$ is measurable.

Because f is measurable on each E_i.

- ⇨ $\bigcup_i \{x \in E_i: f(x) > \alpha\}$ is measurable.

⇨ {x∈ E| f(x) $>$ α} is measurable.
Hence f is measurable on E.

2.6 Theorem: If f is measurable function on E and $E_1 \subseteq E$ is measurable set then f is a measurable function on E_1.

Proof: Let α be any real number.

Consider the set {x∈ E_1| f(x) $>$ α} = {x∈ E| f(x) $>$ α}∩ E_1 is measurable.

2.7 Theorem. If f and g are measurable functions on a common domain E, then the set A= {x∈ E: f(x)< g(x)} is measurable.

Proof. For each rational number r, define

A_r = { x∈ E: f(x) < r < g(x)}

Or we can write

A_r = { x∈ E: f(x)< r}∩ {x∈ E: g(x)> r}

Since f and g are measurable on E, so the two sets on R.H.S. are measurable sets is measurable.

Now, we observe that

{x∈E: f(x) <g(x)} = $\cup_{r\in Q} A_r$

Since the rationals are countable, so A is countable union of measurable sets and so is measurable.

This proves the theorem.

2.8 Theorem. A continuous function defined on a measurable set is measurable.

Proof. Let f be a continuous function defined on measurable set E. Let α be any real number. We now claim that {x ∈ E : f(x) ≥ α } is closed.

Let A= { x ∈ E: f(x) ≥ α } (1)

To prove that A is closed, it is sufficient to show that $A' \subseteq A$. (2)

A' being derived set of A.

Let $x_0 \in A'$ be arbitrary element. Then $x_0 \in A'$ implies x_0 is limit point of A.

It implies that there exist a sequence $\{x_n\}$ whose elements $x_n \in A$ such that

$$\lim_{n\to\infty} x_n = x_0$$

Moreover, f is continuous at x_0; it follows that by definition of continuity $x_n \to x_0$ implies $f(x_n) \to f(x_0)$ (3)

By (2), we see that x_n ∈ A for all n ∈ N.

⇒ f(x_n) ≥ α for all n∈ N.

⇒ $\lim_{n\to\infty} f x_n \geq \alpha$

$\Rightarrow f(x_0) \geq \alpha$ by virtue of (3)

$\Rightarrow x_0 \in A$ by (1)

Further any $x_0 \in A'$ implies $x_0 \in A$

$\Rightarrow A' \subseteq A$

⇒ A is closed

⇒A is measurable

⇒{x ∈ E: f(x) ≥ α } is measurable.

Hence a continuous function f is measurable on E.

Converse of above theorem is not true, that is, A measurable function need not be continuous.

2.9 Example. Consider a function f: R → [0,1]

defined by f(x) = $\begin{cases} \mathbf{1} & \boldsymbol{if\ 0 \leq x \leq 1} \\ \mathbf{0} & \boldsymbol{if\ otherwise} \end{cases}$ **.**

Clearly function is not continuous since 0 is the point of discontinuity.

For any real number α,

$$\{x \in N: f(x) > \alpha \} = \begin{cases} \varphi, & \alpha \geq 1 \\ R, & \alpha < 0 \\ [0,1), & \alpha \leq 0 < 1 \end{cases}$$

Since R, φ, [0,1) are measurable implies f is measurable function on E.

2.10 Theorem. Let f be a function defined on a measurable set E then f is measurable iff for any open set G in R the inverse image f^{-1} (G) is measurable set.

Proof. Let f be a measurable function and let G be any set in R. Since every open sets can be written as countable union of disjoint open intervals.

Suppose, G = $\cup_n I_n = \cup_n (a_n, b_n)$

Then $f^{-1}(G) = f^{-1}(\cup_n I_n) = \cup_n \{ f^{-1}(I_n)\}$

$$= \cup_n \{ x : f(x) \in I_n\}$$

$$= \cup_n \{x : f(x) \in (a_n, b_n)\}$$

but $\{x : f(x) \in (a_n, b_n)\} = \{x : a_n < f(x) < b_n\}$

$$= \{x : f(x) > a_n\} \cap \{x : f(x) < b_n\}$$

Since f is measurable function. So both sets on R.H.S. are measurable and hence

$\{x : f(x) \in (a_n, b_n)\}$ is measurable.

Again f^{-1} (G) is measurable. [since countable union of measurable sets is measurable]

Conversely:- Let f^{-1} (G) be measurable for every open set G in R. We have to prove that f is measurable function.

Take G = (α, ∞) where α is any real no.

Then $f^{-1}(\alpha, \infty)$ is measurable

that is, $\{x : f(x) \in (\alpha, \infty)\}$ is measurable

that is , $\{x : f(x) > \alpha\}$ is measurable.

Thus f is measurable function.

2.11 Theorem. Let f be continuous and g be measurable function then fog is measurable.

Proof. Let α be any real number then

$$\{x : fog(x) > \alpha\} = \{x : f(g(x)) > \alpha\}$$
$$= \{x : f(g(x)) \in (\alpha, \infty)\}$$
$$= \{x : g(x) \in f^{-1}(\alpha, \infty)\}.$$

Now, (α, ∞) is open subset of R and f is continuous implies $f^{-1}(\alpha, \infty)$ is open set.

Hence, it can be written as countable union of disjoint open intervals say

$$f^{-1}(\alpha, \infty) = \cup_n I_n = \cup_n (a_n, b_n).$$

Therefore,

$$\{x : fog(x) > \alpha\} = \{x : g(x) \in \cup_n I_n\} = g^{-1}(\cup_n I_n)$$
$$= \cup_n g^{-1}(I_n)$$
$$= \cup_n \{x : g(x) \in I_n\}$$
$$= \cup_n \{x : g(x) \in (a_n, b_n)\}$$
$$= \cup_n \{x : a_n < g(x) < b_n)\}$$
$$= \cup_n \{x : g(x) > a_n)\} \cap \cup_n \{x : g(x) < b_n)\}.$$

Since g is measurable function. Both sets on R.H.S. are measurable and their intersection is measurable. Also countable union of measurable sets is measurable. Hence the result.

2.12 Definition. A function f is said to be a step function iff

f(x) = Ci , $\xi_{i-1} < x < \xi_i$ for some subdivision of [a, b] and some constants Ci .

Example: A function f : [0, 1] → R defined as f(x) = $\begin{cases} \alpha, & a \le x \le c \\ \beta, & c \le x \le b \end{cases}$ where α, β are constant, f is a step function.

Remark: Every step function is a measurable function.

2.13 Theorem. For any real no c and two measurable real- valued functions, f and g, the functions f +c, cf, f+g, f-g, fg and f/g (g$\neq$ 0), $|f|$ are all measurable.

Proof. We are given that f is measurable function and c ais any real number. Then for any real number α

$$\{x|\ f(x) + c > \alpha\} = \{x|\ f(x) > \alpha - c\}$$

But $\{x|\ f(x) > \alpha - c\}$ is measurable by the condition (a) of the definition. Hence

$\{x|\ f(x) + c > \alpha\}$ and so | f(x) $+c$ is measurable.we next consider the function cf . in case c= 0, cf is the constant function 0 and hence is measurable since every constant function is continuous and so measurable. In case c $>$ 0 we have

$$\{x|\ cf(x) > \alpha\} = \{x|\ f(x) > \frac{\alpha}{c}\}, \text{ and so measurable.}$$

In case c $<$ 0, we have $\{x|\ cf(x) > r\} = \{x|\ f(x) < \frac{r}{c}\}$ and so measurable.

Now if f and g are two measurable real valued functions defined on the same domain, we shall show that f+g is measurable. To show that it is sufficient to show that the set

$\{x|\ f(x)+g(x) > \alpha\}$ is measurable.

if f(x) +g(x)$>$ α, then | f(x) $> \alpha - g(x)$ and by he cor. of the axiom of Archimedes there is a rational number r such that $\alpha - g(x) < r < f(x)$

since the functions f and g are measurable , the sets $\{x|\ f(x) > r\}$ and $\{x|\ f(x) > \alpha - r\}$ are measurable. Therefore, there intersection $S_r = \{x|\ f(x) > \alpha - c\} \cap \{x|\ f(x) > \alpha - r\}$ also measurable.

It can be shown that $\{x|f(x)+g(x) > \alpha\} = \cup \{S_r \mid r \text{ is rational}\}$

Since the set of rational is countable and countable union of measurable sets is measurable , the set $\cup \{S_r \mid r \text{ is rational}\}$ and hence $\{x|f(x)+g(x) > \alpha\}$is measurable which proves that

f(x) +g(x) is measurable. From this part it follows that f- g = f (-g) is also measurable, since when g is measurable (-g) is also measurable. Next we consider fg.

The measurability of fg follows that from the identity fg = $\frac{1}{2}[(f+g)^2 - f^2 - g^2]$, if we prove that f^2 is measurable when f is measurable. For this it is sufficient to prove that

$\{x|\in E|f^2(x) > \alpha\}$, α is real number, is measurable.

Let α be a negative real number. Then it is clear that the set $\{x|f^2(x) > \alpha\}$ = E (domain of the measurable function f). But E is measurable by the definition of f. Hence $\{x|f^2(x) > \alpha\}$ is measurable when $\alpha < 0$.

Now let $\alpha \geq 0$, then $\{x|f^2(x) > \alpha\} = \{x|\ f(x) > \sqrt{\alpha}\} \cup \{x|\ f(x) < -\sqrt{\alpha}\}$.

Since f is measurable, it follows from this equality that $\{x|f^2(x) > \alpha\}$ is measurable for $\alpha \geq 0$.

Hence f^2 is also measurable when f is also measurable. Therefore, the theorem follows from the above identity, since measurability of f and g imply the measurability of f+g.

Consider $\frac{f}{g}(g \neq 0) = f.\frac{1}{g}$

First we have to prove that $\frac{1}{g}$ is measurable.

Consider the set $\left\{x : \left(\frac{1}{g}\right)(x) > \alpha\right\} = \left\{x : \frac{1}{g(x)} > \alpha\right\}$

$$= \begin{cases} \{x: g(x) > 0\} if\ \alpha = 0 \\ \{x: g(x) > 0\} \cap \left\{x: g(x) < \frac{1}{\alpha}\right\} if\ \alpha > 0 \\ \{x: g(x) > 0\} \cup \left\{\{x: g(x) > 0\} \cap \left\{x: g(x) < \frac{1}{\alpha}\right\}\right\} if\ \alpha < 0 \end{cases}$$

Since g is measurable in each case ,i.e., $\left\{x : \left(\frac{1}{g}\right)(x) > \alpha\right\}$is measurable.

⇨ $\frac{1}{g}$ is measurable.

Since f and $\frac{1}{g}$ are measurable.

⇨ $\frac{f}{g}$ is measurable.

Now If f is measurable then |f| is also measurable.

It suffices to prove that measurability of the set {x| f(x) $>\ \alpha\}$= E (domain of f)

But E is assumed to be measurable. Hence {x| f(x) $>\ \alpha\}$={x| f(x) $>\ \alpha\}\cup$ {x| f(x)$<\ -\alpha\ \}$

The right hand side of the equality is measurable since f is measurable. Hence {x| f(x) $> \alpha\}$is measurable. Hence |f| is measurable.

2.14 Remark: Converse of (vii) is not true.

Example: Let P be a non-measurable subset of [0, 1) = E

Define a function f : E→ R as

f(x) = $\begin{cases} \mathbf{1} \quad \boldsymbol{if\ x \in P} \\ \mathbf{-1} \quad \boldsymbol{if\ x\ not\ belongs\ to\ P} \end{cases}$

⇨ f is not measurable because $\{x: f(x) > 0\} = \{x: f(x) = 1\} = P\ which\ is\ non-measurable.$

Also, for any real $\alpha, \{x: |f|(x) > \alpha\} = \{x: |f(x)| > \alpha\} = \begin{cases} \varphi, if\ \alpha \geq 1 \\ E, if\ \alpha < 1 \end{cases}$

Since E and φ are measurable.

⇨ $\{x: |f|(x) > \alpha\}\ is\ measurable.$

2.15 Theorem. Let $\{f_n\}_{n=1}^{\infty}$ **be a sequence of measurable functions. Then sup{f_1,f_2,…,f_n}, inf {f_1,f_2,…,f_n},** $\sup_n, \inf_n, \lim_n f_n and \lim f_n$ **are measurable.**

Proof. Define a function M(x) = $sup\{f_1(x), f_2(x), \ldots, f_n(x)\}$ we shall show that

{x| M(x)> $\alpha\}$is measurable. In fact {x| M(x)> $\alpha\}$= $\cup_i^n\{x: f_i(x) > \alpha\}$

Since each f_i is measurable, each of the set $\{x|\ f_i(x) > \alpha\}$is measurable and therefore their union is also measurable. Hence $\{x|\ M(x) > \alpha\}$ and so M(x) is measurable. Similarly we define the function m(x) = inf $\{f_1, f_2,,,, f_n\}$, since $\{x|\ m(x) < \alpha\} = \cup_i^n \{x: f_i(x) < \alpha\}$ and

since $\{x|\ f_i(x) < \alpha\}$is measurable on account of the the measurability of f_i, it follows that $\{x|\ m(x) < \alpha\}$ and so m(x) is measurable . Define a function M'(x) $= \sup_n f_n(x) = \sup\{f_1, f_2,,,, f_n\}$

We shall show that the set $\{x|\ M'(x) > \alpha\}$ is measurable for any real α.

Now $\{x|\ M'(x) > \alpha\} = \cup_{n=1}^{\infty} \{x: f_n(x) > \alpha\}$ is measurable, since each f_n is measurable.

Similarly if we define m'(x) $= \inf_n f_n(x)$, then $\{x|\ m'(x) < \alpha\} = \cup_{n=1}^{\infty} \{x: f_n(x) < \alpha\}$ and therefore measurability of f_n implies that of m'(x). Now since $\overline{lim} f_n = \inf_n \sup_{k \geq n} f_k$ and

$\underline{lim} f_n = \sup_n \inf_{k \geq n} f_k$, the upper and lower limit are measurable.

2.16 Corollary: If $\{f_n\}$ is a sequence of measurable functions converging to f. Then f is also measurable.

Proof: Since$\{f_n\}$ converges to f ,i.e., $\lim_{n \to \infty} f_n = f$

Then $\overline{lim} f_n = \underline{lim} f_n = \lim_{n \to \infty} f_n$

i.e., f = $\overline{lim} f_n = \underline{lim} f_n$

Hence f is measurable because $\overline{lim} f_n$ *and* $\underline{lim} f_n$ *are measurable.*

2.17 Corollary: The set of points on which a sequence $\{f_n\}$ of measurable functions converges is measurable.

Proof: By above theorem $\overline{lim} f_n$ *and* $\underline{lim} f_n$ *are measurable.*

⇨ $\overline{lim} f_n - \underline{lim} f_n$ *is measurable.*

Therefore, $\{x: [\overline{lim} f_n - \underline{lim} f_n](x) = \alpha\}$ *is measurable* $\forall \alpha$.

In Particular, for $\alpha = c$

$$\{x: [\overline{lim} f_n - \underline{lim} f_n](x) = 0\} \text{ is measurable.}$$

i.e.,

$$\{x: [\overline{lim} f_n(x) = \underline{lim} f_n(x)]\} \text{ is measurable.}$$

i.e., set of these points for which$\{f_n\}$converges is measurable.

2.18 Definition. Let f and g be measurable functions. Then we define

f^+= Max (f, 0)

f^- = Max (-f, 0)

f $\vee$g = $\frac{f+g+|f-g|}{2}$ i.e. Max (f, g) and

f $\wedge$g = $\frac{f+g-|f-g|}{2}$ i.e. min (f, g)

2.19 Theorem. Let f be a measurable function. Then $\overset{+}{f}$ and f^{-} are both measurable.

Proof. Let us suppose that f > 0 . Then we have

$$f = \overset{+}{f} - f^{-} \qquad \text{(i)}$$

Now let us take f to be negative.

Then $\overset{+}{f} =$ Max (f, 0) = 0, (ii)

$\overset{+}{f} =$ Max (-f, 0) = -f

Therefore on subtraction $f = \overset{+}{f} - f^{-}$

In casef = 0 , then $\overset{+}{f} = 0$, $f^{-} = 0$. (iii)

Therefore $f = \overset{+}{f} - f^{-}$

Thus for all f we have, $f = \overset{+}{f} - f^{-}$

Also adding the components of (i) we have

$$f = |f| = \overset{+}{f} + f^{-} \qquad \text{(v)}$$

since f is positive. And from (ii) when f is negative we have

$$\overset{+}{f} + f^{-} = 0 - f^{-} = f^{-} = |f| \qquad \text{(vi)}$$

In case f is zero, then

$$\overset{+}{f} + f^{-} = 0 + 0 = 0 = |f| \qquad \text{(vii)}$$

That is for all f, we have

$$|f| = \overset{+}{f} + f^{-} \qquad \text{(viii)}$$

Adding (iv) and (viii) we have $f + |f| = 2\overset{+}{f}$,

$$\overset{+}{f} = \frac{1}{2}(f + |f|) \qquad \text{(ix)}$$

Similarly on subtracting we obtain $\overset{-}{f} = \frac{1}{2}(f - |f|)$ (x)

Since measurability of f implies the measurability of | f | it is obvious from (ix) and (x) that $\overset{+}{f}$ and $\overset{-}{f}$ are measurable.

2.20 Theorem. If f and g are two measurable functions, then f $\vee$g and f $\wedge$g are measurable.

Proof. We know that

$$f \vee g = \frac{f+g+|f-g|}{2}$$

$$f \wedge g = \frac{f+g-|f-g|}{2}$$

Now measurability of f _ measurability of |f|. Also if f and g are measurable, then f+g, f-g are measurable. Hence f$\vee$g and f $\wedge$g are measurable.

2.21 Definition. Characteristic function of a set E is defined by $\chi_{E(X)} = \begin{Bmatrix} 1, x \in E \\ 0, x \notin E \end{Bmatrix}$

This is also known as indicator function**.**

2.22 Examples of measurable function

Example. Let E be a set of rationals in [0,1]. Then the characteristic function $\chi_{E(X)} = \begin{Bmatrix} 1, x \in E \\ 0, x \notin E \end{Bmatrix}$ is measurable.

Proof. For the set of rationals in the given interval, we have $\chi_{E(X)} = \begin{Bmatrix} 1, x \in E \\ 0, x \notin E \end{Bmatrix}$

It is sufficient to prove that { x | $\chi_{E(X)} > \alpha\}$ is measurable for any real α.

Let us suppose first $that\ \alpha \geq 1$. Then { x | $\chi_{E(X)} > \alpha\}$ ={ x | $\chi_{E(X)} > 1\}$

Hence the set { x | $\chi_{E(X)} > \alpha\}$ is empty in this very case. But outer measure of any empty set is zero. Hence for $\alpha \geq 1$, the set { x | $\chi_{E(X)} > \alpha\}$ and so $\chi_{E(X)}$ is measurable .

Further let 0$\leq \alpha \leq 1$. Then { x | $\chi_{E(X)} > \alpha\} = \mathrm{E}$

But E is countable and therefore measurable. Hence $\chi_{E(X)}$ is measurable.

Lastly, let$\alpha \leq 0$. Then { x | $\chi_{E(X)} > \alpha\}$ = [0,1] and therefore measurable. Hence the result.

2.23 Theorem. Characteristic function χ_A is measurable if and only if A is measurable.

Proof. Let A be measurable. Then $\chi_A(x) = \begin{Bmatrix} 1, x \in A \\ 0, x \notin A \end{Bmatrix}$

Hence it is clear from the definition that domain of χ_A is A $\cup$A^c which is measurable due to the measurability of A. Therefore, if we prove that the set {x | $\chi_{A(x)} > \alpha$} is measurable for any real α, we are through.

Let $\alpha \geq 0$. Then {x | $\chi_{A(x)} >$ }= {x | $\chi_{A(x)} = 1$}=A(by the definition of Characteristic function.)

But A is given to be measurable. Hence for $\alpha \geq 0$. The set {x | $\chi_{A(x)} > \alpha$ } is measurable.

Now let us take $\alpha < 0$. Then {x | $\chi_{A(x)} >$ }= $A \cup A^C$

Hence {x | $\chi_{A(x)} >$ } is measurable for $\alpha < 0$ also, since $A \cup A^C$ has been proved to be measurable. Hence if A is measurable, then χ_A is also measurable. Conversely, let us suppose that $\chi_{A(x)}$ is measurable. That is,

the set {x | $\chi_{A(x)} > \alpha$ } is measurable for any real α.

Let $\alpha \geq 0$. Then {x | $\chi_{A(x)} >$ }= {x | $\chi_{A(x)} = 1$}= A

Therefore, measurability of{x | $\chi_{A(x)} >$ } implies that of the set A for $\alpha \geq 0$. Now consider $\alpha < 0$. Then {x | $\chi_{A(x)} >$ }= $A \cup A^C$

Thus measurability $\chi_{A(x)}$ of implies measurability of the set $A \cup A^C$ which imply A is measurable.

2.24 Simple Function: Let f be a real valued function defined on X. If the range of f is finite. We say that f is a simple function.

Let E⊆X and put $\chi_{E(X)} = \begin{Bmatrix} 1, x \in E \\ 0, x \notin E \end{Bmatrix}$

Suppose the range of f consists of the distinct number $c_1, c_2, \ldots, c_n$.

Let $E_i = \{x : f(x) = c_i\} (i = 1, 2, \ldots, n)$

Then f = $\sum_{i=1}^{n} c_i \, \chi_{E_i}$

i.e., every simple function is a finite linear combination of characteristic function. It is clear that f is measurable if and only if the sets $E_1, E_2, \ldots, E_n$ are measurable.

2.25 Remarks:

1. **Every step function is a simple function.**
2. **Every simple function is measurable.**
   Proof: Let f be a simple function defined as above.
   Then we have
   f(x) = $\sum_{i=1}^{n} c_i \, \chi_{E_i}$(x)
   $= c_1\chi_{E_1}(x) + c_2\chi_{E_2}(x) + \cdots + c_n\chi_{E_n}(x)$

$$\therefore f(x) = c_1, x \in E_1$$
$$f(x) = c_2, x \in E_2$$
$$\therefore f(x) = c_i, x \in E_i$$

$$\therefore E_i = \{x: f(x) = c_i\}$$

Since each E_i is measurable. Thus χ_{E_i} is measurable because χ_A is measurable if and only if A is measurable.

Hence f is measurable.

3. **Characteristic function of measurable set is a simple function.**
4. **Product of the simple function and finite linear combination of simple functions is again a simple function.**

2.26 Theorem. (Approximation Theorem). For every non-negative measurable function f, there exists a non-negative non-decreasing sequence $\{f_n\}$ of simple functions such that $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in E$

In the general case if we do not assume non-negativeness of f , then we say For every measurable function f, there exists a sequence $\{f_n\}$, $n \in N$ of simple function which converges (pointwise) to f . i.e. "Every measurable function can be approximated by a sequence of simple functions."

Proof. Let us assume that $f(x) \geq 0$ and $x \in E$. Construct a sequence

$$f_n(x) = \begin{Bmatrix} \frac{i-1}{2^n}, for\ \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} for\ i = 1,2\ \ , n2^n \\ n, \quad f(x) \geq n \end{Bmatrix} \text{ for every } n \in N.$$

If we take n = 1, then

$$f_1(x) = \begin{Bmatrix} \frac{i-1}{2}, for\ \frac{i-1}{2} \leq f(x) < \frac{i}{2} for\ i = 1,2 \\ 1, \quad f(x) \geq 1 \end{Bmatrix}$$

That is, $$f_1(x) = \begin{Bmatrix} 0, for\ 0 \leq f(x) < \frac{1}{2} \\ \frac{1}{2}, \quad for\ \frac{1}{2} \leq f(x) < 1 \\ 1\ \ for\ f(x) \geq 1 \end{Bmatrix}$$

Similarly taking n = 2, we obtain

$$f_2(x) = \begin{Bmatrix} \frac{i-1}{4}, for\ \frac{i-1}{4} \leq f(x) < \frac{i}{4}\ for\ i = 1,2,,,8 \\ 2, \quad f(x) \geq 2 \end{Bmatrix}$$

That is,

$$f_2(x) = \begin{Bmatrix} 0\ for\ \ 0 \leq f(x) < \frac{1}{4} \\ \frac{1}{4}\ for\ \frac{1}{4}\ \leq f(x) < \frac{1}{2} \\ \dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots.. \\ \frac{7}{4}\ for\ \frac{7}{4}\ \leq f(x) < 2 \\ 2\ for\ f(x) \geq 2 \end{Bmatrix}$$

Similarly we can write $f_3(x)$ (x) and so on. Clearly all f_n are positive whenever f is positive and also it is clear that $f_n \leq f_{n+1}$. Moreover f_n takes only a finite number of values. Therefore $\{f_n\}$ is a sequence of non-negative, non decreasing functions which assume only a finite number of values.

Let us denote

$$E_{ni} = f^{-1}\left[\frac{i-1}{n}, \frac{i}{n}\right] = \left\{x \in E \mid \frac{i-1}{2} \leq f(x) < \frac{i}{2}\right\}$$

and

$$E_n = f^{-1}[n.\infty) = \{x \in E \mid f(x) \geq n\}$$

Both of them are measurable. Let

$$f_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n}\chi_{E_{n_i}} + n\chi_{E_n} \quad \text{for every } n \in N .$$

Now $\sum_{i=1}^{n2^n} \frac{i-1}{2^n}\chi_{E_{n_i}}$ is measurable, since $\chi_{E_{n_i}}$ has been shown to be measurable and characteristic function of a measurable set is measurable. Similarly χ_{E_n} is also measurable since E_n is measurable. Hence each f_n is measurable. Now we prove the convergence of this sequence.

Let $f(x) < \infty$. That is f is bounded. Then for some n we have

$$\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}$$

$$\frac{i-1}{2^n} - \frac{i-1}{2^n} \leq f(x) - \frac{i-1}{2^n} < \frac{i}{2^n}$$

$\Rightarrow \quad 0 \leq f(x) - \frac{i-1}{2^n} < \frac{i}{2^n}$

$\Rightarrow \quad 0 \leq f(x) - f_n(x) < \frac{i}{2^n}$ (by the def of $f_n(x)$)

$\Rightarrow \quad f(x) \leq f_n(x) < \varepsilon$

or $|f(x) - f_n(x)| \leq \frac{1}{2^n} < \varepsilon \forall n \geq m \text{ and } x \in E.$

since m does not depend upon point.

Therefore, convergence is uniform.

Let us suppose now that f is not bounded. Then $f(x) = \infty$

$\Rightarrow$ $f(x) \geq n \text{ for every } n \in N$

But $f_n(x) = n$

$\Rightarrow$ $\lim_{n\to\infty} f_n(x) = \infty = f(x).$

When **we do not assume non-negativeness of the function** then since we know that $\overset{+}{f}$ and $\overset{-}{f}$ are both non-negative, we have by what we have proved above

$$\overset{+}{f} = \lim_{n\to\infty} \phi_n'(x) \qquad \text{(i)}$$

$$\overset{-}{f} = \lim_{n\to\infty} \phi_n''(x) \qquad \text{(ii)}$$

where $\emptyset_n'(x)$ and $\emptyset_n''(x)$ are simple functions. Also we have proved already that

$$f = \overset{+}{f} - f^-$$

Now from (i) and (ii) we have

$$\overset{+}{f} - f^- = \lim_{n\to\infty} \phi_n'(x) - \lim_{n\to\infty} \phi_n''(x)$$

$$= \lim_{n\to\infty} (\phi_n'(x) - \phi_n''(x))$$

$$= \lim_{n\to\infty} \phi_n(x)$$

(since the difference of two simple functions is again a simple function). Hence the theorem.

We now introduce the terminology "almost everywhere" which will be frequently used in the Sequel.

2.27 Definition. A statement is said to hold almost everywhere in E if and only if it holds everywhere in E except possibly at a subset D of measure zero.

(a) Two functions f and g defined on E are said to be equal almost everywhere in E iff f(x) =g(x) everywhere except a subset D of E of measure zero.

(b) A function defined on E is said to be continuous almost everywhere in E if and only if there exists a subset D of E of measure zero such that f is continuous at every point of E□D.

2.28 Theorem. (a) If f is a measurable function on the set E and $E_1 \subseteq E$is measured set, then f is a measurable function on E_1.

(b) If f is a measurable function on each of the sets in a countable collection{E_i } of disjoint measurable sets, then f is measurable.

Proof. (a) For any real α, we have {x ∈ E_1, f(x) > α } = { x ∈E; f(x) > α } ∩E1. The result follows as the set on the right-hand side is measurable.

(b)Write E = $\bigcup_{i=1}^{\infty} E_i$, Clearly, E, being the union of measurable set is measurable. The result now follows, since for each real α, we have

E = { x ∈E: f(x) > α } = { x ∈$\bigcup_{i=1}^{\infty} E_i$: f(x) > α }

2.29 Theorem. Let f and g be any two functions which are equal almost everywhere in E. If f is measurable so is g.

Proof. Since f is measurable, for any real, the set {x | f(x) > } is measurable. We shall show that the set {x | g(x) > } is measurable. To do so we put

E_1 = {x | f(x) > } and E_2 = {x | g(x) > α }. Consider the sets

$E_1 - E_2$ and $E_2 - E_1$.

These are subsets of $\{x: f(x) \neq g(x)\}\begin{bmatrix} \because x \in E_1 - E_2 \Longrightarrow x \in E_1 and\ x \notin E_2 \\ f(x) > \alpha, g(x) \ngtr \alpha \Longrightarrow f(x) \neq g(x) \end{bmatrix}$

But f = g a.e.

⇨ m$\{x: f(x) \neq g(x)\}$ = 0

$$E_1 - E_2 \subseteq \{x: f(x) \neq g(x)\}$$

m$(E_1 - E_2) \leq$ m$\{x: f(x) \neq g(x)\}$ = 0

⇨ m$(E_1 - E_2) \leq 0$ But m$(E_1 - E_2) \geq 0$

⇨ m$(E_1 - E_2) = 0$

Similarly m$(E_2 - E_1)$ =0

∴ m$(E_2 - E_1)$ = 0 = m$(E_1 - E_2)$

⇨ $(E_1 - E_2)$ and $(E_2 - E_1)$ are measurable.

⇨ $E_2 = [E_1 \cup (E_2 - E_1)] - (E_1 - E_2)$

Since E_1, $E_2 - E_1$ and $(E_1 - E_2)^C$ are measurable therefore it follows that E2 is measurable. Hence the theorem is proved.

2.30. Corollary. Let $\{f_n\}$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n = f$ almost everywhere. Then f is a measurable function.

Proof. We have already proved that if $\{f_n\}$ is a sequence of measurable functions then $\lim_{n\to\infty} f_n$ is measurable. Also, it is given that $\lim_{n\to\infty} f_n$ = f a.e. Therefore, using the above theorem, it follows that f is measurable.

2.31 Definition: (Restriction of f to E_1)

Let f be a function defined on E, then the function f_1 defined on E_1 contained in E .i.e., $E_1 \subseteq E$ by $f_1(x) = f(x)$, $x \in E_1$ is called restriction of f to E_1 and denoted by f/ E_1.

2.32 Exercise : Let f be a measurable function defined on E, then its restriction to E_1 is also measurable where E_1 is a measurable subset of E.

Solution : Let f_1 = f/ E_1 i.e., f_1 is restriction of f to E_1.

Let α *be any real number*.

$\{x \in E_1: f_1(x) > \alpha\} = \{x \in E_1: f(x) > \alpha\}$ $[\because f_1 = f \text{ on } E_1]$

= { x ∈E: f(x) > α } ∩ $E_1$is measurable on E and E_1 is also measurable and intersection of measurable sets is measurable. Hence f_1 is measurable on E_1.

2.33Exercise:Let f be a measurable function defined on $\boldsymbol{where\ E_1\ and\ E_2\ are\ measurable.}$ $\boldsymbol{Then\ the\ function\ f\ is\ measurable\ on\ E_1UE_2\ iff\ \frac{f}{E_1} and \frac{f}{E_2}\ are\ measurable.}$

Solution: Let f_1 = f/ E_1 and Let f_2 = f/ E_2

Let E = E_1UE_2

Clearly E is measurable because $E_1\ and\ E_2\ are\ measurable$. Suppose f is measurable on E then by previous exercise f_1 is measurable on E_1 and f_2 is measurable on E_2.

Conversely, Let $\alpha\ be\ any\ real\ number$.

Therefore

{x ∈E: f(x) > α } = { x ∈ E_1UE_2 : f(x) > α }

= { x ∈E_1: f(x) > α }∪ { x ∈E_2: f(x) > α }

= { x ∈E_1: f_1(x) > α }∪ { x ∈E_2: f_2(x) > α }

because f_1 is measurable on E_1 and f_2 is measurable on E_2.

- ⇨ f is measurable on E = E_1UE_2.

2.24 Theorem. If a function f is continuous almost everywhere in E, then f is measurable.

Proof. Since f is continuous almost everywhere in E, there exists a subset D of E with m*D = 0 such that f is continuous at every point of the set C = E-D.

To prove that f is measurable, let α denote any given real number.

Consider the set {x ∈E | f(x) > } = B(say)

We have to show that B is measurable. If B ∩C = $\varphi, then\ B \subseteq D$.

- ⇨ m*(B) ≤ m*(D) = 0.
- ⇨ m*(B) = 0.
- ⇨ B is measurable.

Now suppose that B ∩C ≠ φ. For this purpose, let x denote an arbitrary point in B ∩C. Then x ∈ $B\ and\ x \in C$ ⟹f(x) > α and f is continuous at x. Hence there exists an open interval U_x containing x such that f(y) > α hold for every point y of E∩ Ux. Let U =$\bigcup_{x\in B\cap C} U_x$.

Since x ∈E ∩Ux ⊂B holds for every x ∈B ∩C, we have

B ∩C⊂E ∩Ux ⊂B. This implies B = (E ∩U) ∪(B ∩D). As an open subset of R, U is measurable. Hence E ∩U is measurable. On the other hand, since m*(B∩D) ≤m*D = 0 ,B∩D is also measurable. This implies that B is measurable. This completes the proof of the theorem.

2.25 Littlewood's three principles of measurability

The following three principles concerning measure are due to Littlewood.

First Principle. Every measurable set is a finite union of intervals.

Second Principle. Every measurable function is almost a continuous function.

Third Principle. If $\{f_n\}$ is a sequence of measurable function defined on a set E of finite measure and if $f_n(x) \to f(x)$ on E, then $f_n(x)$ converges almost uniformly on E.

First of all we consider third principle. We shall prove Egoroff's theorem which is a slight modification of third principle of Littlewood's.

2.26 Theorem. Let E be a measurable set with finite measure and $\{f_n\}$ be a sequence of measurable functions defined on a set E such that

$$\mathbf{f_n(x) \to f(x) \text{ for each } x \in E.}$$

Then given $\varepsilon > 0$ and $\delta > 0$, there corresponds a measurable subset A of E with m(A) < δ and an integer N such that $|f_n(x) - f(x)| < \varepsilon \; \forall \; x \in E - A \text{ and } n \geq N$.

Proof: Consider the sets $G_n = \{x \in E: |f_n(x) - f(x)| \geq \varepsilon\}$

Now since f_n and f are measurable.

So the sets $G_n{}'s$ are also measurable.

Now define $E_k = \bigcup_{n=k}^{\infty} G_n$.

$= \{x: x \in G_n \; for \; some \; n \geq k\}$

$= \{x: x \in E, |f_n(x) - f(x)| \geq \varepsilon \; for \; some \; n \geq k\}$

We observe that $E_{k+1} \subseteq E_k$.

On the contrary, we assume that for each $x \in E_k \forall \; k$.

Then for any fixed given k, we must have

$$E_k = \{|f_n(x) - f(x)| \geq \varepsilon \; for \; some \; n \geq k\}$$

But this leads to $f_n(x) \nrightarrow f(x)$. a contradiction.

Hence for each $x \in E$ there is some E_k such that $x \notin E_k \Longrightarrow \bigcap_{k=1}^{\infty} E_k = \emptyset$

Now measure of E is finite, so by proposition of decreasing sequence, we have

$$\lim_{n\to\infty} m(E_n) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = m(\emptyset) = 0$$

$$\lim_{n\to\infty} m(E_n) = 0.$$

Hence given $\delta > 0$, $\exists$ an integer N such that m(E_k) $< \delta \forall \; k \geq N$.

In particular put k = N

$$m(E_n) < \delta$$

$$m\{x: x \in E, |f_n(x) - f(x)| \geq \varepsilon for \; some \; n \geq N\} < \delta$$

If we write A = E_n, then m(A) $< \delta \; and$

E-A = $\{x: x \in E, |f_n(x) - f(x)| < \varepsilon for \; all \; n \geq N\}$

In other words,

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N \text{ and } x \in E - A$$

This completes the proof.

2.27 Definition: A Sequence {fn} of functions defined on a set E is said to converge almost everywhere to f if $\lim_{n\to\infty} f_n(x) = f(x)\ \forall\, x \in E - E_1$ *where* $E_1 \subset E$,

$$m(E_1) = 0.$$

2.28 Theorem. Let E be a measurable set with finite measure and {fn} be a sequence of measurable functions converging almost everywhere to a real valued function f defined on a set E. Then given $\boldsymbol{\varepsilon > 0}$ and $\boldsymbol{\delta} > 0$, there corresponds a measurable subset A of E with m(A) < $\boldsymbol{\delta}$ and an integer N such that $|\boldsymbol{f_n(x) - f(x)}| < \varepsilon\ \forall\, x \in E - A \text{ and } n \geq N$.

Proof: Let F be a set of points of E for which $f_n(x) \nrightarrow$f. Then m(F) = 0.

Since f_n (x) →f(x) almost everywhere, then

f_n (x) →f(x) $\forall\, x \in E - F = E_1 (say)$

Now applying the last theorem for the set E_1, we get a set $A_1 \subseteq E_1$ with $m(A_1) < \delta$ and an integer N such that $|f_n(x) - f(x)| < \varepsilon\ \forall\, x \in E_1 - A_1 \text{ and } n \geq N$.

Now the required result follows if we take

$$A = A_1 \cup F \text{ as shown below.}$$

$$m(A) = m(A_1 \cup F) = m(A_1) + m(F) = m(A_1) + 0 = m(A_1) < \delta$$

Also $E - A = E - (A_1 \cup F) = E \cap (A_1 \cup F)^c$

$= E \cap {A_1}^c \cap F^c = (E \cap F^c) \cap {A_1}^c$

$= (E - F) \cap {A_1}^c = E_1 \cap {A_1}^c = E_1 - A_1$

i.e., E-A = $E_1 - A_1$

Hence we have found a set A ⊆ E with m(A) < δ and an integer N such that $|f_n(x) - f(x)| < \varepsilon\ \forall\, x \in E - A \text{ and } n \geq N$.

2.29 Definition: A Sequence {fn} of functions is said to converge almost uniformly everywhere to a measurable function f defined on a measurable set E if for each ε >0, ∃ *a measurable set* A ⊆ E with m(A) < ε such that and an integer N such that f_n *converges to* f *uniformly on* $E - A$.

2.30 Theorem.(Egoroff's Theorem). Let {fn} be a sequence of measurable functions defined on a set E of finite measure such that f_n (x) →f(x) almost everywhere. Then to each η> 0 there corresponds a measurable subset A of E such that m (A) < η such that fn(x) converges to f(x) uniformly on E-A.

Proof. Applying last theorem with $\varepsilon = 1, \delta = \eta/2$

We get a measurable subset $A_1 \subseteq E$ with m $(A_1) < \eta/2$ and positive integer N_1 such that

$$|f_n(x) - f(x)| < 1 \text{ for all } n \geq N_1 \text{ and } x \in E_1 (= E - A_1)$$

Again taking $\varepsilon = \frac{1}{2}, \delta = \eta/2^2$

We get another measurable subset $A_2 \subseteq E_1$ with m $(A_2) < \eta/_{2^2}$ and positive integer N_2 such that

$$|f_n(x) - f(x)| < \frac{1}{2} \; for \; all \; n \geq N_2 \; and \; x \in E_2(= E_1 - A_2)$$

Continuing like that at kth stage, we get a measurable subset $A_k \subseteq E_{k-1}$ with

m $(A_k) < \eta/_{2^k}$ and positive integer N_k such that

$$|f_n(x) - f(x)| < \frac{1}{k} \; for \; all \; n \geq N_k \; and \; x \in E_k(= E_{k-1} - A_k)$$

Now we set A = $\bigcup_{k=1}^{\infty} A_k$

Then we have

m(A) $\leq \sum_{k=1}^{\infty} m(A_k) < \sum_{k=1}^{\infty} \eta/_{2^k} = \eta . \sum_{k=1}^{\infty} \frac{1}{2^k} = \eta.$

Also E-A = E- $\bigcup_k A_k = \bigcap_k [E_{k-1} - A_k] = \bigcap_k E_k [\because E_{k-1} - A_k = E_k]$

Let $x \in E - A, then \; x \in E_k \forall \; k \; and \; so \; |f_n(x) - f(x)| < \frac{1}{k} \forall n \geq N_k.$

Choose k such that $\frac{1}{k} < \varepsilon \; so \; that \; we \; get$

$|f_n(x) - f(x)| < \varepsilon \; \forall \; x \in E - A \; and \; n \geq N_k = N.$

This completes the proof of the theorem.

Now we pass to the **second principle of Littlewood**. This is nothing but approximation of measurable functions by continuous functions. In this connection we shall prove the following theorem known as Lusin Theorem after the name of a Russian Mathematician Lusin, N.N.

2.31 Lusin Theorem: Let f be a measurable real valued function defined on closed interval [a,b], then given $\boldsymbol{\delta} > 0, \exists \; a \; continuous \; function \; g \; on \; [\boldsymbol{a}, \boldsymbol{b}] \boldsymbol{such \; that}$

$$\boldsymbol{m}\{\boldsymbol{x}: \boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{g}(\boldsymbol{x})\} < \delta.$$

Proof: First we prove two lemmas.

Lemma 1. Let F be a closed subset of R, then a function g: F→ $\boldsymbol{R}$ is continuous if sets $\{\boldsymbol{x}: \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{a}\} \; \boldsymbol{and} \; \{\boldsymbol{x}: \boldsymbol{g}(\boldsymbol{x}) \geq \boldsymbol{b}\}$ are closed subsets of F for every rational a and b.

Proof: Let $\{x: g(x) \leq a\} \; and \; \{x: g(x) \geq b\} \; are \; closed \; subset \; of \; F.$

$\Rightarrow \{x: g(x) > a\} \cap \{x: g(x) < b\}$ is open subset of F.

.i.e., $\{x: a < g(x) < b\}$ is open.

i.e., $\{x: g(x) \in (a, b)\} \; is \; open \; in \; F.$

i.e., $g^{-1}(a, b) is \; open \; in \; F.$

Let O be any open set in R then O can be written as countable union of disjoint open intervals with rational end points.

Let O = $\bigcup_{n=1}^{\infty}(a_n, b_n)$

Then g^{-1}O = $g^{-1}(\bigcup_{n=1}^{\infty}(a_n, b_n))$ = $\bigcup_{n=1}^{\infty} g^{-1}(a_n, b_n)$

Since $g^{-1}(a,b) \, is \; open$ and countable union of open set is open.

$\Rightarrow g^{-1}(O) \; is \; open \Rightarrow g \; is \; continuous.$

Lemma 2. Let f: [a, b] $\to \boldsymbol{R}$ be a measurable function, then given $\boldsymbol{\delta} > 0, \exists \; a \; closed \; subset \; F \; of \; E = [\boldsymbol{a}, \boldsymbol{b}] \boldsymbol{such \; that \; m}(\boldsymbol{E} - \boldsymbol{F}) < \; \delta \; and \frac{\boldsymbol{f}}{\boldsymbol{F}} \boldsymbol{is \; continuous}.$

Proof: Let $\{r_n\} \; be \; a \; sequence \; of \; all \; rational \; numbers.$

For n$\in N$, take $A_n = \{x : f(x) \geq r_n\}$

And $A_n{}^* = \{x : f(x) \leq r_n\}$

Clearly each $A_n \; and A_n{}^* \; are \; measurable \; [\because \; f \; is \; measurable]$

Then $\exists$ closed sets $B_n \subset A_n \; and \; B_n{}^* \subset A_n{}^* \; such \; that$

$$m(A_n - B_n) < \frac{\delta}{2^n.3} \; and \; m(A_n{}^* - B_n{}^*) < \frac{\delta}{2^n.3}$$

Let $D = [\bigcup_{n=1}^{\infty}(A_n - B_n)] \cup [\bigcup_{n=1}^{\infty}(A_n{}^* - B_n{}^*)]$

Clearly D is measurable.

Therefore m(D) $\leq \sum_{n=1}^{\infty} m(A_n - B_n) + \sum_{n=1}^{\infty} m(A_n{}^* - B_n{}^*)$

m(D) < $\sum_{n=1}^{\infty} \frac{\delta}{2^n.3} + \sum_{n=1}^{\infty} \frac{\delta}{2^n.3}$

$= \frac{\delta}{3} + \frac{\delta}{3} = \frac{2\delta}{3}$

$$\Rightarrow m(D) < \frac{2\delta}{3}.$$

Now E and D are measurable.

- ⇨ E-D is measurable.

  Then for given $\delta > 0, \exists \; a \; closed \; set \; F \subseteq E - D \; such \; that \; m(E - D - F) < \frac{\delta}{3}$

  Now E-F = D$\cup$ $(E - F - D)$

- ⇨ m(E-F) = m(D) + m$(E - F - D) < \frac{2\delta}{3} + \frac{\delta}{3} = \; \delta$

  Let h = f/F

  To show that h is continuous on F.

  For rational number r_n,

$$\{x : h(x) \leq r_n\} = \{x : f(x) \leq r_n\} \cap F$$
$$= A_n{}^* \cap F = \left[\left((A_n{}^* - B_n{}^*) \cup B_n{}^*\right)\right] \cap F$$
$$= \left[\left((A_n{}^* - B_n{}^*) \cap F\right)\right] \cup [B_n{}^* \cap F]$$
$$= \emptyset \cup [B_n{}^* \cap F]$$
$$= B_n{}^* \cap F$$

$$D = \left[\bigcup_{n=1}^{\infty}(A_n - B_n)\right] \cup \left[\bigcup_{n=1}^{\infty}(A_n{}^* - B_n{}^*)\right]$$

$$\Rightarrow (A_n{}^* - B_n{}^*) \subset D$$

$$\because\ F \subseteq E - D \Rightarrow F \cap D = \emptyset.$$

$$\{x: h(x) \le r_n\} = B_n{}^* \cap F$$

Since $B_n{}^*$ $is\ closed\ in\ E = [a, b]$.

$$B_n{}^* \cap F\ is\ closed\ in\ F.$$

⇨ $\{x: h(x) \le r_n\}\ is\ closed\ in\ F.$

By lemma 1, h is continuous.

So f/F is continuous.

Lusin Theorem:(Proof):- We have

f:[a, b]→ R is measurable function, then by lemma(2), for given $\delta > 0, \exists\ a\ closed\ set\ F \subset E$ $such\ that\ m(E - F) < \delta\ and\ h = \frac{f}{F} is\ continuous.$

Now using result "Every real valued continuous function defined on a closed subset of a real number can be extended continuously to all real numbers."

So h can be extended to continuous function h*: R→ R.

Let g :[a, b]→ $R, g\ is\ continuous$

$$and\ for\ x\ \in F, g(x) = f(x) on\ F.$$

$$and\ \{x \in E: f(x) \ne g(x)\} \subseteq E - F$$

$$m\{x \in E: f(x) \ne g(x)\} \le m(E - F) < \delta.$$

"Convergence in Measure"

The notion of convergence in measure is introduced by F.Reisz and E.Fisher in 1906-07. Sometimes it is also called approximate convergence.

2.32 Definition. A sequence < f_n > of measurable functions is said to convergence in measure to f on a set E, written as $f_n \overset{m}{\to} f\ on\ E,$

If given δ> 0, $\exists\ m \in$ N such that for all n $\ge m$, we have

$m\{x||f(x) - f_n(x)| \ge\ \varepsilon\} <\ \delta.$

Or $\lim\limits_{n\to\infty}\ m\{x||f(x) - f_n(x)| \ge\ \varepsilon\} = 0$

This means that for all sufficiently large value of n, functions f_n of the sequence < fn > differ from the limit function f by a small quantity with the exception of the set of point whose measure is arbitrary small (<δ).

2.33 Theorem: If sequence $\{f_n\}$ converges in measure to the function *f*, then it converges in measure to every function *g* which is equivalent to the function.

Proof: For each $\varepsilon > 0$, we have

$\{x: |f_n(x) - g(x)| \ge \varepsilon\} \subset \{x: f(x) \ne g(x)\} \cup \{x: |f_n(x) - f(x)| \ge \varepsilon\}$

Since g is equivalent to f, then we have

$$m\{x: f(x) \neq g(x)\} = 0.$$

$$m\{x: |f_n(x) - g(x)| \geq \varepsilon\} \leq m\{x: f(x) \neq g(x)\} + m\{x: |f_n(x) - f(x)| \geq \varepsilon\}$$
$$\leq m\{x: |f_n(x) - f(x) \geq \varepsilon|\} < \delta$$

$$\Rightarrow \boldsymbol{f_n \xrightarrow{m} g}.$$

Hence the result.

2.34 Theorem: If sequence $\{f_n\}$ converges in measure to the function *f*, then the limit function *f* is unique a.e.

Proof: Let g be another function such that $\boldsymbol{f_n \xrightarrow{m} g}$.

Since $|f - g| \leq |f - f_n| + |f_n - g|$

Now we observe that for each $\varepsilon > 0$,

$$\{x: |f(x) - g(x)| \geq \varepsilon\} \subset \left\{x: |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x: |f_n(x) - g(x)| \geq \frac{\varepsilon}{2}\right\}$$

Since by proper choice of ε, the measure of both the sets on the right can be made arbitrary small, we have

$m\{x: |f(x) - g(x)| \geq \varepsilon\}$=0

- ⇨ f = g almost everywhere. Hence the proof.

2.35 Theorem: Let $\{f_n\}$ be a sequence of measurable functions which converges to f a.e. on X. Then $f_n \xrightarrow{m} f$ *on X*.

Proof: For each n$\in N$ *and* $\varepsilon > 0$, *Consider the sets*

$$S_n(\varepsilon) = \{x \in X: |f_n(x) - f(x)| \geq \varepsilon\}$$

Let $\delta > 0$ *be any arbitrary number, then* $\exists$*a measurable set* $A \subset X$

With m(A) < δ and the number N such that

$$|f_n(x) - f(x)| < \varepsilon \; \forall x \in X - A \text{ and } n \geq N$$

Then it follows that $S_n(\varepsilon) \subset A \; \forall \; n \geq N$

- ⇨ $m(S_n(\varepsilon)) < m(A) < \delta \forall \; n \geq N$
- ⇨ $\lim_{n\to\infty} m(S_n(\varepsilon)) = 0$

Hence$\boldsymbol{f_n \xrightarrow{m} f}$ ***on X***.

2.36 Remark: The converse of the above theorem need not be true i.e, convergence in measure is more general than a.e. infact there are sequence of measurable functions that converges in measure but fails to converge at any point.

To affect we consider the following example

$$\boldsymbol{f_n: [0, 1] \to R \; as}$$

$$\boldsymbol{f_n(x) = \begin{cases} 1, if \; x \in \left[\frac{k}{2^t}, \frac{k+1}{2^t}\right] \\ 0, otherwise \end{cases}}$$

Let n = $k + 2^t$ *where* $0 \leq k \leq 2^t$.

Let $\varepsilon > 0$ *be given*. Choose an m $\in N$ *such that* $\frac{2}{m} < \varepsilon$

Then $m\{x: |f_n(x) - 0| \geq \varepsilon\} = m\{x: |f_n| \geq \varepsilon\}$

$$=\frac{1}{2^t} < \frac{1}{2^n} \begin{bmatrix} \because n = k + 2^t < 2^t + 2^t \\ < 2.2^t\ ,\ \ \frac{1}{2^t} < \frac{2}{n} \end{bmatrix} \qquad (*)$$

$$\leq \frac{2}{m} < \varepsilon\ \ \forall\, n \geq m$$

- ⇨ f_n converges in measure to zero for x$\in [0,1]$
- ⇨ $i.e., f_n \overset{m}{\rightarrow} [0,1]$

  f_n (x) has value 1 for arbitrary large value of n and so it does not converge to zero a.e. because on taking n very large, we get 2^t large and hence number of subintervals of type (*) increase and possibility of $f_n(x) = 1$ is more.

2.37 Theorem (F. Riesz). "Let < fn > be a sequence of measurable functions which converges in measure to f. Then there is a subsequence < f_{n_k} > of < fn > which converges to **f almost everywhere**."

Proof. Let $\boldsymbol{f_n \overset{m}{\rightarrow} f}$.

Let us consider two sequences $\left\{\frac{1}{n}\right\}$ *and* $\left\{\frac{1}{2^n}\right\}$ *of real numbers such that*

$$\frac{1}{n} \rightarrow 0\ \ as\ n \rightarrow \infty\ as\ \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.$$

We now choose a strictly increasing sequence $\{n_k\}$ of positive integer as follows

Let n_1 be a positive integer such that

$$m\left(\left\{x: \left|f_{n_1}(x) - f(x)\right| \geq 1\right\}\right) < \frac{1}{2}$$

Such a number n_1 exists since in view $\boldsymbol{f_n \overset{m}{\rightarrow} f}$ for a given $\varepsilon_1 = 1 > 0$ *and*

$\delta_1 = \frac{1}{2} > 0, \exists$ *an integer* n_1*such that*

$$m(\{x: |f_n(x) - f(x)| \geq 1\}) < \frac{1}{2} \forall n \geq n_1$$

In particular for n = n_1.

Similarly, Let n_2 be a positive number such that $n_2 \geq n_1$ *and*

$$m\left(\left\{x: \left|f_{n_2}(x) - f(x)\right| \geq \frac{1}{2}\right\}\right) < \frac{1}{2^2}\ and\ so\ on.$$

Continuing in this process, we get the positive number$n_k \geq n_{k-1}$

$$m\left(\left\{x: \left|f_{n_k}(x) - f(x)\right| \geq \frac{1}{k}\right\}\right) < \frac{1}{2^k}$$

Now set $E_k = \bigcup_{i=k}^{\infty} \left\{x: \left|f_{n_i}(x) - f(x)\right| \geq \frac{1}{i}\right\}, k \in N$.

And $E = \bigcap_{k=1}^{\infty} E_k$

Then it is clear that $\{E_k\}$ is decreasing sequence of measurable sets.

Therefore $m(E) = \lim_{k\to\infty} m(E_k)$

But $m(E_k) = m\left\{\bigcup_{i=k}^{\infty} \left\{x: \left|f_{n_i}(x) - f(x)\right| \geq \frac{1}{i}\right\}\right\}$

$$\leq \sum_{i=k}^{\infty} m\left\{x: \left|f_{n_i}(x) - f(x)\right| \geq \frac{1}{i}\right\}$$

$$< \sum_{i=k}^{\infty} \frac{1}{2^i} \to 0 \; as \; k \to \infty$$

$$= \frac{1}{2^{k-1}}$$

Hence m(E) = 0.

Thus it remains to be verified that the sequence $< f_{n_k} >$ converges to f on X-E.

So let $x_0 \notin E$. Then $x_0 \notin E_m$ for some positive integer m.

i.e., $x_0 \notin \left\{x: \left|f_{n_k}(x) - f(x)\right| \geq \frac{1}{k}\right\}, k \geq m$

$$\Rightarrow \left|f_{n_k}(x) - f(x)\right| < \frac{1}{k}, k \geq m$$

But $\frac{1}{k} \to 0$ as k$\to \infty$

Hence $\lim_{k\to\infty} f_{n_k}(x_0) = f(x_0)$.

Since $x_0 \in X - E$ was arbitrary, it follows that

$\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for each x $\in X - E$ and so $\left\{f_{n_k}\right\}$ $converges\ to\ f\ a.e.$

This completes the proof.

SECTION – III

THE LEBESGUE INTEGRAL

Lebesgue integration is an alternative way of defining the integral in terms of measure theory that is used to inte grate a much broader class of functions than the Riemann integral or even the Riemann-Stieltjes integral. The idea behind the Lebesgue integral is that instead of approximating the total area by dividing it into vertical strips, one approximates the total area by dividing it into horizontal strips.

3.1 The shortcomings of the Riemann integral suggested the further investigations in the theory of integration. We give a resume of the Riemann Integral first.

Let f be a bounded real- valued function on the interval [a, b] and let

$a = \xi_0 < \xi_1 < \cdots < \xi_n = b$

Be a partition of [a, b]. Then for each partition we define the sums

$S = \sum_{i=1}^{n}(\xi_i - \xi_{i-1})M_i$

and $s = \sum_{i=1}^{n}(\xi_i - \xi_{i-1})m_i$

where

$M_i = \sup_{\xi_{i-1}<x<\xi_i} f(x)$, $m_i = \inf_{\xi_{i-1}<x<\xi_i} f(x)$

We then define the upper Riemann integral of f by

$R\int_a^b f(x)dx = \inf S$

With the infimum taken over all possible subdivisions of [a, b].

Similarly, we define the lower integral

$R\int_a^b f(x)dx = \sup s.$

The upper integral is always at least as large as the lower integral, and if the two are equal we say that f is Riemann integrable and call this common value the Riemann integral of f. We shall denote it by

$R\int_a^b f(x)$

To distinguish it from the Lebesgue integral, which we shall consider later.

By a **step function** we mean a function ψ which has the form

$$\psi(x) = c_i\, , \xi_{i-1} < x < \xi_i$$

for some subdivision of [a, b] and some set of constants c_i .

The integral of $\psi(x)$ is defined by

$$R\int_a^b \psi(x)\,dx = \sum_{i=1}^{n} c_i\left(\xi_i - \xi_{i-1}\right).$$

With this in mind we see that

$R\overline{\int_a^b} f(x)dx = \inf \int_a^b \psi(x)\,dx$

for all step function $\psi(x) \geq f(x)$.

Similarly,

$R\underline{\int_a^b} f(x)dx = \sup \int_a^b \phi(x)\,dx$

for all step functions $\phi(x) \leq f(x)$.

3.2. Example: If

$f(x) = \begin{cases} 1 \text{ if x is rational} \\ o \text{ if x irrational} \end{cases}$

then $R\overline{\int_a^b} f(x)dx = b - a$ and $R\underline{\int_a^b} f(x)dx = 0$.

Thus we see that f(x) is not integrable in the Riemann sense.

3.3. The Lebesgue Integral of a bounded function over a set of finite measure

The example we have cited just now shows some of shortcomings of the Riemann integral. In particular, we would like a function which is 1 in measurable set and zero elsewhere to be integrable and have its integral the measure of the set.

The function χ_E defined by

$$\chi_E = \begin{cases} 1 \in E \\ o\ x \notin E \end{cases}$$

is called the **characteristic function** on E. A linear combination

$$\phi(x) = \sum_{i=1}^{n} a_i\,\chi_E(x)$$

is called a **simple** function if the sets E_i are measurable. This representation for ϕ is not unique. However, we note that a function ϕ is simple if and only if it is measurable and assume only a finite number of values. If ϕ is simple function and $[a_1, a_2, \ldots, a_n]$ the set of non- zero values of ϕ, then

$\phi = \sum a_i \chi_{A_i}$,

where $A_i = \{\{x|\ \phi(x) = a_i\}$. **This representation for ϕ is called the canonical representation and it** is characterized by the fact that the A_i are disjoint and the a_i distinct and non- zero.

If ϕ vanishes outside a set of finite measure, we define the integral ϕ by

$$\int \phi(x)dx = \sum_{i=1}^{n} a_i mA_i$$

when ϕ has the canonical representation $\phi = \sum_{i=1}^{n} a_i\chi_{A_i}$. we sometimes abbreviate the expression for this integral to $\int \phi$. If E is any measurable set, we define $\int_E \emptyset = \int \phi.\chi_E$

It is often convenient to use representations which are not canonical, and the following lemma is useful.

3.4. Lemma. If $E_1, E_2,\ldots,E_n$ are disjoint measurable subset of E then every linear combination

$\phi = \sum_{i=1}^{n} c_i\chi_{E_i}$

With real coefficients $c_1, c_2,\ldots, c_n$ is a simple function and

$\int \phi = \sum_{i=1}^{n} c_i mE_i$.

Proof. It is clear that ϕ is a simple function. Let $a_1, a_2,\ldots, a_n$ denote the non- zero real number in $\phi(E)$. For each j = 1, 2, …, n . Let

$$A_j = \bigcup_{c_i=a_j} E_i$$

Then we have $A_j = \phi^{-1}(a_j) = \{x \mid \phi(x) = a_j\}$

and the canonical representation

$$\phi = \sum_{j=1}^{n} a_j\chi_{A_j}$$

Consequently, we obtain

$$\begin{aligned} \int \phi &= \sum_{j=1}^{n} a_j mA_j \\ &= \sum_{j=1}^{n} a_j m \quad [\cup_{c_i=a_j} E_i] \\ &= \sum_{j=1}^{n} a_j \sum_{c_i=a_j}^{n} mE_i \quad \text{(Since } E_i \text{ are disjoint, additivity of measures applies)} \end{aligned}$$

$$\sum_{j=1}^{n} c_j mE_i$$

This completes the proof of the theorem.

3.5. **Theorem.** Let ϕ and ψ be simple functions which vanish outside a set of finite measure. Then

$\int (a\phi + b\psi) = a\int \phi + b\int \psi$ and, if $\phi \geq \psi$ a.e., then $\int \phi \geq \int \psi$

Proof. Let $\{A_i\}$ and $\{B_i\}$ be the sets which occur in the canonical representations of ϕ and ψ. Let A_0 and B_0 be the sets where ϕ and ψ are zero. Then the sets E_k obtained by taking all the intersection $A_i \cap B_j$ form a finite disjoint collection of measurable sets, and we write

$$\phi = \sum_{k=1}^{N} a_k\chi_{E_k}$$

$$\psi = \sum_{k=1}^{N} b_k \chi_{E_k}$$

and so

$$a\phi + b\psi = a\sum_{k=1}^{N} a_k \chi_{E_k} + b\sum_{k=1}^{N} b_k \chi_{E_k}$$

$$= \sum_{k=1}^{N} aa_k \chi_{E_k} + \sum_{k=1}^{N} b_k \chi_{E_k}$$

$$= \sum_{k=1}^{N} (aa_k + bb_k) \chi_{E_k}$$

Therefore

$$a\phi + b\psi = \sum_{k=1}^{N} (aa_k + bb_k) mE_k$$

$$= a\sum_{k=1}^{N} a_k mE_k + b\sum_{k=1}^{N} b_k mE_k$$

$$= a\int \phi + b\int \psi\,.$$

To prove the second statement, we note that

$$\int \phi - \int \psi = \int \phi - \psi \geq 0\,,$$

Since the integral of **a simple function which is greater than or equal to zero almost everywhere is non- negative** by the definition of the integral.

3.6. Remark. We know that for any simple function ϕ we have

$$\phi = \sum_{k=1}^{N} a_i \chi_{E_i}$$

Suppose that this representation is neither canonical nor the sets E_i's are disjoint. Then using the fact that characteristics functions are always simple function we observe that

$$\int \phi = \int a_1 \chi_{E_1} + \int a_2 \chi_{E_2} + \dots + \int a_n \chi_{E_n}$$

$$= a_{1\int \chi_{E_1}} + a_{2\int \chi_{E_2}} + \cdots + a_{n\int \chi_{E_n}}$$

$$= a_1 mE_1 + a_2 mE_2 + \cdots + a_n mE_n$$

$$= \sum_{k=1}^{N} a_i mE_i$$

Hence for any representation of ϕ, we have

$$\int \phi = \sum_{k=1}^{N} a_i mE_i$$

Let f be a bounded real valued function and E be a measurable set of finite measure. By analogy with the Riemann integral we consider for simple functions ϕ and ψ the numbers

$$\inf_{\psi \geq f} \int_E \psi$$

and

$$\sup_{\phi \leq f} \int_E \phi$$

and ask when **these two numbers are equal.** The answer is given by the following proposition .

3.7. Theorem. Let f be defined and bounded on a measurable set E with mE finite. In order that

$$\inf_{f \leq \psi} \int_E \psi\,(x)dx = \sup_{f \geq \psi} \int_E \psi(x)dx$$

For all simple functions ϕ and ψ, it is necessary and sufficient that f be measurable.

Proof. Let f be bounded by M and suppose that f is measurable. Then the sets

$$E_k = \left\{x \middle| \frac{KM}{n} \geq f(x) > \frac{(K-1)M}{n}\right\}, -n \leq K \leq n\,,$$

Are measurable, disjoint and have union E. Thus

$$\sum_{k=-n}^{n} mE_k = mE$$

The simple function defined by

$$\psi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} k\chi_{E_k}(x)$$

and

$$\phi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\chi_{E_k}(x)$$

satisfy

$$\phi_n(x) \leq f(x) \leq \psi_n(x)$$

Thus $\quad \inf \int_E \psi(x)dx \leq \int_E \psi_n(x)dx = \frac{M}{n}\sum_{k=-n}^{n} kmE_k$

and $\quad \sup \int_E \phi(x)dx \geq \int_E \phi_n(x)dx = \frac{M}{n}\sum_{k=-n}^{n}(k-1)m\,E_k$

hence $\quad 0 \leq \inf \int_E \psi(x)dx - \sup \int_E \phi(x)dx \leq \frac{M}{n}\sum_{k=-n}^{n} mE_k = \frac{M}{n} mE$.

Since n is arbitrary we have

$$\inf \int_E \psi(x)dx - \sup \int_E \phi(x)dx = 0 ,$$

and the condition is sufficient.

Suppose now that $\inf\limits_{\psi \geq f} \int_E \psi(x)dx = \sup\limits_{\phi \leq f} \int_E \phi(x)dx$.

Then given n there are simple functions ϕ_n and ψ_n such that

$$\phi_n(x) \leq f(x) \leq \psi_n(x)$$

And $\quad (1) \quad \int \psi_n(x)dx - \int \phi_n(x)dx < \frac{1}{n}$

Then the functions $\quad \psi^* = \inf \psi_n$

And $\quad \phi^* = \sup \phi_n$

Are measurable and $\quad \phi^*(x) \leq f(x) \leq \psi^*(x)$.

Now the set

$$\Delta = \{x|\ \phi^*(x) < \psi^*(x)\}$$

is the union of the sets

$$\Delta_v = \left\{x \middle|\ \phi^*(x) < \psi^*(x) - \frac{1}{v}\right\}.$$

But each Δ_v is contained in the set $\left\{x \middle| \phi_n(x) < \psi_n(x) - \frac{1}{v}\right\}$, and this latter set by (1) has measure less than $\frac{v}{n}$. Since n is arbitrary, $m\Delta_v = 0$ and so $m\Delta = 0$. Thus $\phi^* = \psi^*$ except on a set of measure zero, and $\phi^* = f$ except on a set of measure zero. Thus f is measurable and the condition is also necessary.

3.8. **Definition.** If f is a bounded measurable function defined on a measurable set E with mE finite, we define the Lebesgue integral of f over E by

$$\int_E f(x)dx = \inf \int_E \psi(x)$$

for all simple functions $\psi \geq f$.

By previous theorem, this may also be defined as

$$\int_E f(x)dx = \sup \int_E \phi(x)$$

for all simple functions $\phi \leq f$.

We sometime write the integral as $\int_E f$. If E =[a,b] we write $\int_a^b f$ instead of $\int_{[a,b]} f$.

Definition and existence of the Lebesgue integral for bounded functions

3.9. Definition. Let F be a bounded function on E and let E_k be a subset of E. Then we define $M[f, E_k]$ and $m[f, E_k]$ as

$$M[f, E_k] = \underset{x\varepsilon E_k}{l.u.b}\, f(x)$$

$$m[f, E_k] = \underset{x\varepsilon E_k}{g.l.b}\, f(x)$$

3.10. Definition. By a **measurable partition of E we mean a finite collection** $P = \{E_1, E_2, \ldots, E_n\}$ of measurable subsets of E such that

$$\bigcup_{k=1}^{n} E_k = E$$

And such that $m(E_j \cap E_k) = 0$ $(j, k = 1,2, \ldots, n, j \neq k)$

The sets $E_1, E_2, \ldots, E_n$ are called the **components of P.**

If P and Q are measurable partitions, then Q is called a refinement of P if every component of Q is wholly contained in some component of P.

Thus a measurable partition P is a finite collection of subsets whose union is all of E and whose intersections with one another have measure zero.

3.11. Definition. Let f be a bounded function on E and let $P=\{E_1, E_2, \ldots, E_n\}$ be any measurable partition E. we define the upper sum U[f, P] as

$$U[f;\ P] = \sum_{k=1}^{n} M[f; E_k].\, mE_k$$

Similarly, we define the lower sum L[f; P] as

$$L[f;\ P] = \sum_{k=1}^{n} m[f; E_k].\, mE_k$$

As in the case of Riemann integral, we can see that every upper sum for f is greater than or equal to every lower sum for f.

We then define the Lebesgue upper and lower integral of a bounded function f on E by

$$\inf_P U[f;\ P] \text{ and } \sup_P L[f;\ P]$$

Respectively taken over all measurable position of E. We denote them respectively by

$$\overline{\int_E} f \text{ and } \underline{\int_E} f$$

3.12. Definition. We say that a bounded function f on E is Lebesgue integrable on E if

$$\overline{\int_E} f \text{ and } \underline{\int_E} f$$

Also we know that if ψ is a simple function, then

$$\int_E \psi = \sum_{k=1}^{n} a_k mE_k$$

Keeping this in mind, we see that

$$\overline{\int_E} f = \inf \overline{\int_E} \psi(x)dx$$

For all simple functions $\psi(x) \geq f(x)$. Similarly

$$\underline{\int_E} f = \sup \underline{\int_E} \phi(x)dx$$

For all simple functions $\phi(x) \leq f(x)$.

Now we use the theorem :

" Let f be defined and bounded on a measurable set E with mE finite. In order that

$$\inf_{f\leq\psi} \int_E \psi(x)dx = \sup_{f\geq\phi} \int_E \phi(x)dx$$

for all simple functions ϕ and ψ, it is necessary and sufficient that f is measurable."

And our definition of **Lebesgue integration** takes the form :

" If f is a bounded measurable function defined on a measurable set E with mE finite , we define the (Lebesgue) integral of f over E by

$$\int_E f(x)dx = \inf \int_E \psi(x)dx$$

for all simple functions $\psi \geq f$."

The following theorem shows that the Lebesgue integral is in fact a generalization of the Riemann integral.

3.13. Theorem. Let f be a bounded function defined on [a,b]. If f is Riemann integrable on [a, b], then it is measurable and

$$R\int_a^b f(x)dx = \int_a^b f(x)$$

Proof . Since f is a bounded function defined on [a, b] and is Riemann integrable, therefore,

$$R\int_a^{\overline{b}} f(x)dx = \inf_{\phi\geq f}\int_a^b \phi(x)dx$$

and

$$R\int_{\underline{a}}^b f(x)dx = \sup_{\psi\leq f}\int_a^b \psi(x)dx$$

for all step functions ϕ and ψ and then

$$R\int_a^{\overline{b}} f(x)dx = R\int_{\underline{a}}^b f(x)dx$$

$$\Rightarrow \inf_{\phi\geq f}\int_a^b \phi(x)dx = \sup_{\psi\leq f}\int_a^b \psi(x)dx \qquad \text{(i)}$$

Since every step function is a simple function, we have

$$R\int_{\underline{a}}^b f(x)dx = \sup_{\psi\leq f}\int_a^b \psi(x)dx \leq \inf_{\phi\geq f}\int_a^b \phi(x)dx = R\int_a^{\overline{b}} f(x)dx$$

Then (i) implies that

$$\sup_{\psi\leq f}\int_a^b \psi(x)dx = \inf_{\phi\geq f}\int_a^b \phi(x)dx$$

and this implies that f is measurable also.

3.14. Comparison of Lebesgue and Riemann integration

(1) The most obvious difference is that in Lebesgue's definition we divide up the interval into subsets while in the case of Riemann we divide it into subintervals.

(2) In both Riemann's and Lebesgue's definitions we have upper and lower sums which tend to limits. In Riemann case the two integrals are not necessarily the same and the function is integrable only if they are same. In the Lebesgue case the two integrals are necessarily the same, their equality being consequence of the assumption that the function is measurable.

(3) Lebesgue's definition is more general than Riemann. We know that if function is the R- integrable then it is Lebesgue integrable also, but the converse need not be true. For example the characteristic function of the set of irrational points have Lebesgue integral but is not R- integrable.

Let χ be the characteristic function of the irrational numbers in [0,1]. Let E_1 be the set of irrational number in [0,1], and let E_2 be the set of rational number in [0,1]. Then $P = [E_1, E_2]$ is a measurable partition of (0,1]. Moreover, χ is identically 1 on E_1 and χ is identically 0 on E_2 . Hence $M[\chi, E_1] = m[\chi, E_2] = 1$, while $M[\chi, E_1] = m[\chi, E_2] = 0$. Hence $U[\chi, P] = 1.m\, E_1 + 0.m\, E_2 = 1$. Similarly $L[\chi, P] = 1.m\, E_1 + 0.M\, E_2 = 1$. Therefore, $U[\chi, P] = L[\chi, P]$.

For Riemann integration

$M[\chi,J] = 1$, $m[\chi,J] = 0$

for any interval $J \subset [0,1]$

$$\therefore U[\chi, J] = 1, L[\chi, J] = 0 .$$

$\therefore$ The function is not Riemann- integrable.

3.15. Theorem. If f and g are bounded measurable functions defined on a set E of finite measure, then

(i) $\int_E af = a \int_E f$

(ii) $\int_E (f+g) = \int_E f + \int_E g$

(iii) If $f \le g$ a.e., then $\int_E f \le \int_E g$

(iv) If $f = g$ a.e., then $\int_E f = \int_E g$

(v) If $A \le f(x) \le B$, then $AmE \le \int_E f \le BmE.$

(vi) If A and B are disjoint measurable set of finite measure, then $\int_{A\cup B} f = \int_A g + \int_B f$

Proof. We know that if ψ is a simple function then so is a ψ.

Hence $\int_E af = \inf\limits_{\psi \ge f} \int_E a\psi = a \inf\limits_{\psi \ge f} \int_E \psi = a \int_E f$

Which proves (i).

To prove (ii) let ε denote any positive real number. These are simple functions $\phi \le f, \psi \ge f, \xi \le g$ and $\eta \ge g$ satisfying

$$\int_E \phi(x)dx > \int_E f - \varepsilon, \qquad \int_E \psi(x)dx < \int_E f + \varepsilon,$$

$$\int_E \xi(x)dx > \int_E g - \varepsilon, \qquad \int_E \eta(x) < \int_E g + \varepsilon,$$

Since $\phi + \xi \le f + g \le \psi + \eta$, we have

$$\int_E (f+g) \ge \int_E (\phi + \xi) = \int_E \phi + \int_E \xi > \int_E f + \int_E g - 2\varepsilon$$

$$\int_E (f+g) \le \int_E (\psi + \eta) = \int_E \psi + \int_E \eta < \int_E f + \int_E g + 2\varepsilon$$

Since these hold for every $\varepsilon > 0$, we have

$$\int_E (f+g) = \int_E f + \int_E g$$

To prove (iii) it suffices to establish

$$\int_E g - f \ \geq 0$$

For every simple function $\psi \geq g - f$, we have $\psi \geq 0$ almost everywhere in E. This means that $\int_E \psi \geq 0$

Hence we obtain

$$\int_E (g-f) = \inf_{\psi \geq (g-f)} \int_E \psi(x) \geq 0 \qquad (1)$$

Which establishes (iii).

Similarly we can show that

$$\int_E (g-f) = \sup_{\psi \leq (g-f)} \int_E \psi(x) \leq 0 \qquad (2)$$

Therefore, from (1) and (2) the result (iv) follows.

To prove (v) we are given that

$$A \leq f(x) \leq B$$

Applying (iv) we are given that

$$\int_E f(x)dx \leq \int_E Bdx = B \int_E dx = BmE$$

That is, $\int_E f \leq BmE$

Similarly we can prove that $\int_E f \geq BmE.$

Now we prove (vi).

We know that $\chi_{A \cup B} = \chi_A + \chi_B$

Therefore , $\int_{A \cup B} f = \int_{A \cup B} \chi_{A \cup B} f = \int_{A \cup B} f(\chi_A + \chi_B)$

$$= \int_{A \cup B} f\chi_A + \int_{A \cup B} f\chi_B$$

$$= \int_A f + \int_B f$$

Which proves the theorem.

3.16. Corollary. If f and g are bounded measurable function then

If $f(x) \geq 0$ on E then $\int_E f \geq 0$ and

If $f(x) \leq 0$ on E then $\int_E f \leq 0$.

Proof : Let ψ be a simple function such that $\psi \geq f$

Since $f(x) \geq 0$ on E $\Rightarrow \psi \geq 0$ on E

$\Rightarrow \int_E \psi \geq 0 \Rightarrow \operatorname*{Inf}_{\psi \geq f} \int_E \psi \geq 0$ i.e. $\int_E f \geq 0$

Similarly, Let ϕ be a simple function such that $\phi \leq f$. Since $f(x) \geq 0$ on E

$\Rightarrow \phi \leq 0$ on E

$\Rightarrow \int_E \phi \leq 0 \Rightarrow \operatorname*{Sup}_{\phi \leq f} \int_E \phi \leq 0$ i.e. $\int_E f \leq 0$

3.17. Corollary. If m(E) = 0 , then $\int_E f = 0$

Or

Integrals over set of measure zero are zero.

Proof : Since f is bounded on E so there exist constant A and B such that

$$A \leq f(x) \leq B$$

$$\Rightarrow A.m(E) \leq \int_E f(x)dx \leq B.m(E) \quad \forall x \in E$$

Since m(E) = 0 $\Rightarrow \int_E f = 0$

3.18. Corollary. If f(x) = k a.e. on E then $\int_E f = k.m(E)$. In particular if f = 0 a.e. on E then $\int_E f = 0$

Proof : Since f(x) = k a.e on E then $\int_E f = 0$

3.19. Corollary. If f = g a.e then $\int_E f = \int_E g$ but converse is not true.

Proof : consider the functions

$f : [-1,1] \longrightarrow R$ and $g : [1,1] \longrightarrow R$

as $f(x) = \begin{cases} 2 \text{ if } x \leq 0 \\ 0 \text{ if } x > 0 \end{cases}$ and g(x) =1 $\forall$ x

Clearly f and g are bounded and measurable functions.

$\Rightarrow$ f and g are lebesgue integrable on [-1,1]

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx$$

$$= \int_{-1}^{0} 2dx + \int_{0}^{1} 0.dx = 2$$

$$\int_{-1}^{1} g(x) = \int_{-1}^{1} 1dx = 1.\,m([-1,1]) = 1.2 = 2$$

Therefore $\int_E f = \int_E g$

But f $\neq$ g a.e on [-1,1]

$$\{\because m\{x \in [-1,1]\,; f \neq g\} = 2 \neq 0\}$$

Therefore f $\neq$ g a.e on E

f and g are not equal even at a single point of [-1,1] as these are defined.

3.20. Corollary. If f = 0 a.e on E then $\int_E f = 0$ but converse is not true.

Proof : Consider the function f : [-1,1] → R as $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx$$

$$= -1 + 1 = 0$$

Clearly f $\neq$ 0 a.e as $m\{x \in [-1,1]\,; f \neq 0\} = m[-1,1] = 2 \neq 0$

So converse is not true.

3.21.Corollary. If $\int_E f = 0$ and $f \geq 0$ on E then f = 0 a.e.

Proof : Suppose E has a subset A where f(x) > 0,

i.e. $A = \bigcup_{x=1}^{\infty} \left\{x \in E\,; f(x) > \frac{1}{n}\right\}$

Let $E_1(n) = \left\{x \in E\ ; f(x) > \frac{1}{n}\right\}$

If possible, suppose there is a positive integer N such that $m(E_1(N)) > 0$.

Then $\int_E f \geq \int_{E_1(N)} f \geq \frac{1}{N}\, m(E_1(N)) > 0$

Which contradicts the fact that $\int_E f = 0$

Thus, $m(E_1(n)) = 0$ for all n≥ 1.

This proves the corollary.

3.22. Corollary. Let f be a bounded measurable function on a set of finite measure E. Then

$\left|\int_E f\right| \leq \int_E |f|$

Proof : The function |f| is measurable and bounded

Now -|f| $\leq$ f $\leq$ |f| on E

By the linearity and monotonicity of integration,

$-\left|\int_E f\right| \leq \int_E f \leq \int_E |f|$

$\Rightarrow \left|\int_E f\right| \leq \int_E |f|$

3.23. The Monotone Convergence Theorem

Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e on E, then $\lim_{n\to\infty} \int_E f_n = \int_E f$

Proof : Since $\{f_n\}$ is an increasing sequence

So $f_n \leq f$ a.e $\forall$ n

$\Rightarrow \overline{\lim} \int f_n \leq \int f$ … (1)

Now by Fatou's Lemma $\int f \leq \underline{\lim} \int f_n$ … (2)

From (1) and (2), we have

$$\overline{\lim} \int f_n = \underline{\lim} \int f$$

Hence the result .

Case II If f is a bounded function on E, then theorem is trivially true. Since in this case

$|f(x)| \leq M \; \forall \; x \in E$ for some number M and thus $\in > 0$, one can choose a $\delta = \left(\frac{\in}{M}\right) > 0$ for which m(A) $< \delta$, then $\int_A f \leq M \int_A 1 = M.\, m(A) < \in$.

3.24. Remark : The technique used in above theorem helps us to evaluate the lebesgue integral of non-negative bounded and unbounded functions.

3.25. Example : Evaluate the Lebesgue integral of the function f : [0,1] → R

$$f(x) = \begin{cases} 1/x^{1/3} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Clearly f is unbounded, non-negative function defined on [0,1]. Now define a sequence of functions $\{f_n\}$ on [0,1] as

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } n < f(x) \end{cases}$$

$$\text{i.e. } f_n(x) = \begin{cases} f(x) & \text{if } x \geq \frac{1}{n^3} \\ n & \text{if } x < \frac{1}{n^3} \end{cases}$$

Clearly $\{f_n\}$ is increasing sequence of non-negative measurable functions such that $f_n \to f$. So by monotone convergence theorem

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\int_0^1 f_n(x)dx$$

$$= \lim_{n\to\infty}\left[\int_0^{1/n^3} f_n(x)dx + \int_{1/n^3}^1 f_n(x)dx\right]$$

$$= \lim_{n\to\infty}\left[\int_0^{1/n^3} ndx + \int_{1/n^3}^1 f(x)dx\right]$$

$$= \lim_{n\to\infty}\left[[nx]_0^{1/n^3} + \int_{1/n^3}^1 x^{-1/3}dx\right]$$

$$= \lim_{n\to\infty}\left[n.\frac{1}{n^3} + \frac{3}{2}\left(1-\frac{1}{n^2}\right)\right]$$

$$= 0 + \frac{3}{2} = \frac{3}{2}$$

3.26. Theorem(Lebesgue Bounded Convergence Theorem). Let $< f_n >$ be a sequence of measurable functions defined on a set E of finite measure and suppose that $< f_n >$ is uniformly bounded, that is , there exist a real number M such that $|f_n(x)| \le M$ for all n ε N and for all x ε E . If $\lim_{n\to\infty} f_n(x) = f(x)$ for each x in E, then

$$\int_E f = \lim_{n\to\infty}\int_E f_n \ .$$

Proof. We shall apply Egoroff's theorem to prove this theorem. Accordingly for a given $\varepsilon > 0$, there is an N and a measurable set $E_0 \subset E$ such that $mE_0^c < \frac{\epsilon}{4M}$ and for $n \ge N$ and x ε E_0 we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{2m(E)}$$

$$\left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right| \le \int_E |f_n - f|$$

$$= \int_{E_0} |f_n - f| + \int_{E_0^c} |f_n - f|$$

$$< \frac{\epsilon}{2m(E)}.m(E_0) + \frac{\epsilon}{4M}2M$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence

$$\int_E f_n \to \int_E f$$

3.27. Remark : Bounded Convergence Theorem need not be true in Riemann integral .

3.28. Example : Let $\{r_i\}$ be a sequence of all rational numbers in [0,1].

Define $S_n = \{r_i : i = 1,2,\dots,n\}, n \in N$

and for each $n \in N$, consider the function $f_n(x) = \begin{cases} 1 \text{ if } x \in S_n \\ 0 \text{ if } x \notin S_n \end{cases} = \{r_1, r_2, \dots, r_n\}$

clearly each f_n is bounded, also f_n is discontinuous at n-points in [0,1] namely points of S_n i.e., $r_1, r_2, \dots, r_n$.

At $x = r_1$

$$\lim_{x\to r_1^-} f_n(x) \neq f_n(r_1) \neq \lim_{x\to r_1^+} f_n(x)$$

Hence Riemann integrable on [0,1]

$[\because \text{A function is Riemann integrable, if it is continous except at a finite number of discontinuity}]$

Now we have proved that

$$\lim_{n\to\infty} R\int f_n(x)dx \neq R\int_0^1 \lim_{n\to\infty} f_n(x)dx$$

$$\Rightarrow R\int_0^1 f_n(x)dx = \int_0^1 f_n(x)dx = \int_{S_n\cup S_n^c} f_n(x)dx$$

$$\{\because S_n \cup S_n^c = [0,1]\}$$

$$= \int_{S_n} f_n(x)dx + \int_{S_n^c} f_n(x)dx$$

$$= \int_{S_n} 1\,dx + \int_{S_n^c} 0\,dx = 1.m(S_n) = 0$$

$$[\because \{S_n\}\text{sequence of rationals } m(S_n) = 0]$$

$$\Rightarrow \lim_{n\to\infty} R\int_0^1 f_n(x)dx = 0$$

Clearly $\{f_n\}$ is convergent to f when f is defined as

$$f(x) = \begin{cases} 1 \text{ if f is rational in } [0,1] \\ 0 \text{ if f is irrational in } [0,1] \end{cases}$$

and f is not Riemann – integrable on [0,1]

$$\Rightarrow R\int_0^1 f(x)dx \text{ does not exists .}$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq R\int_0^1 \lim_{n\to\infty} f_n(x)dx$$

So bounded convergence theorem does not hold in Riemann integral .

The integral of a non-negative function

3.29. Definition. If f is a non-negative measurable function defined on a measurable set E, we define

$$\int_E f = \sup_{h\le f}\int_E h\,,$$

Where h is a bounded measurable function such that $m\{x|h(x) \neq 0\}$ is finite.

3.30. Theorem. If f and g are non-negative measurable functions, then

(i) $\int_E cf = c\int_E f > 0$

(ii) $\int_E (f+g) = \int_E f + \int_E g$ and

(iii) $\text{If } f \le g \text{ a.e., then}$

$$\int_E f \le \int_E g$$

Proof. The proof of (i) and (iii) follow directly from the theorem concerning properties of the integrals of bdd functions.

We prove (ii) in detail.

If $h(x) \le f(x)$ and $k(x) \le g(x)$, we have $h(x) + k(x) \le f(x) + g(x)$, and so

$$\int_E (h+k) \le \int_E (f+g)$$

i.e. $\int_E h + \int_E k \le \int_E (f+g)$.

Taking suprema, we have

(iv) $\int_E f + \int_E g \le \int_E (f+g)$

On the other hand, let ℓ **be a bounded measurable function** which vanishes outside a set finite measure and which is not greater than $(f+g)$. Then we define the functions h and k by setting

$$h(x) = \min(f(x), \ell(x))$$

and $$k(x) = \ell(x) - h(x)$$

we have $$h(x) \le f(x),$$

$$k(x) \le g(x)$$

while h and k are bounded by the bound ℓ and vanish where ℓ vanishes. Hence

$$\int_E \ell = \int_E h + \int_E k \leq \int_E f + \int_E g$$

And so taking supremum, we have

$$\sup_{\ell \leq f+g} \int_E \ell \leq \int_E f + \int_E g$$

That is,

(v) $\int_E f + \int_E g \geq \int_E (f+g)$

From (iv) and (v), we have

$$\int_E (f+g) = \int_E f + \int_E g$$

3.31. Fatou's lemma. If $< f_n >$is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then

$$\int_E f \leq \underline{\lim} \int_E f_n$$

Proof. Let h be a bounded measurable function which is not greater than f and which vanishes outside a set E' of finite measure. Define a function h_n by setting

$h_n(x) = \min\{h(x), f_n(x)\}$

Then h_n is bounded but bounds for h and vanishes outside E' . Now $h_n(x) \to h(x)$ for each x in E' .

Therefore by "**Bounded Convergence theorem"** we have

$$\int_E h = \int_{E'} h = \lim \int_{E'} h_n \leq \underline{\lim} \int_E f_n$$

Taking the supremum over h, we get

$$\int_E f \leq \underline{\lim} \int_E f_n$$

3.32. The inequality in Fatou's lemma may be strict

Consider a sequence $\{f_n\}$ defined on R as

$$f_n(x) = \begin{cases} 1 \text{ if } x \in [n, n+1] & E_1 \\ 0 \quad \text{otherwise} & E_2 \end{cases}$$

Clearly sequence $\{f_n\}$ is sequence of non – negative measurable funcyions defined on R and

$\lim_{n\to\infty} f_n = f$ where $f = 0 \Rightarrow \int_R f = 0$

Also $\int_R f_n = \int_{E_1 \cup E_2} f_n = \int_{E_1} f_n + \int_{E_2} f_n$

$= \int_{E_1} 1 + 0 = m(E_1) = 1$

$\Rightarrow \underline{\lim} \int_R f_n = 1$ and we know that $0 < 1$

So $\int_R f < \underline{\lim} \int_R f_n$

3.33. Fatou's lemma need not good unless the function f_n is non – negative

Let us consider the function $f_n(x) = \begin{cases} -n \text{ if } \frac{1}{n} \le x \le \frac{2}{n} & E_1 \\ 0 \text{ otherwise} & E_2 \end{cases}$

Hence $\lim_{n\to\infty} f_n(x) = f(x) = 0$ a.e $\Rightarrow \int_0^1 f(x)dx = 0$

Also $\int_0^1 f_n(x)dx = \int_{E_1} f_n(x)dx + \int_{E_2} f_n(x)dx$

$= \int_{1/n}^{2/n} -n\, dx + 0 = -1$

Thus $\underline{\lim} \int_0^1 f_n(x)dx = -1$

$$\Rightarrow \int_0^1 f(x)dx \nleq \underline{\lim} \int_0^1 f_n(x)dx$$

3.34. Theorem(Lebesgue Monotone Convergence theorem). Let $< f_n >$ be an increasing sequence of non negative measurable functions and let $f = \lim f_n$. Then

$$\int f = \lim \int f_n$$

Proof. By Fatou's Lemma we have

$$\int f \le \underline{\lim} \int f_n$$

But for each n we have $f_n \le f$, son $\int f_n \le \int f$. But this implies

$$\overline{lim} \int f \le \int f$$

Hence

$$\int f = lim \int f_n$$

3.35. Definition. A non-negative measurable functions f is called integrable over the measurable over the measurable set E if

$$\int_E f < \infty$$

3.36. Theorem. Let f and g be two non-negative measurable functions. If f is integrable over E and g(x)<f(x) on E, then g is also integrable on E, and

$$\int_E (f-g) = \int_E f - \int_E g$$

Proof. Since $\int_E f = \int_E (f-g) + \int_E g$

and the left hand side is finite, the term on the right must also be finite and so g is integrable.

3.37. Theorem. Let f be a non-negative function which is integrable over a set E. The given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have

$\int_A f < \varepsilon$

Proof. If $|f| \le K$, then $\int_A f \le \int_A K = KmA$

Set $\delta < \frac{\epsilon}{K}$ Then $\int_A f < K.\frac{\epsilon}{K} = \epsilon$.

Set $f_n(x) = f(x)$ if $f(x) \le n$ and $f_n(x) = n$ otherwise. Then each f_n is bounded and f_n converges to f at each point. By the monotone convergence theorem there is an N such that $\int_E f_N > \int_E f - \frac{\epsilon}{2}$ *and* $\int_E (f - f_N) < \frac{\epsilon}{2}$.

Choose $\delta < \frac{\epsilon}{2N}$. If mA < δ , we have

$$\int_A f = \int_A (f-f_N) + \int_A f_N$$

$$< \int_E (f - f_N) + NmA$$

$$\left(since \int_A f_N \le \int_A N = NmA\right)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ .$$

3.38. The General Lebesgue Integral

We have already defined the positive part f^+ and negative part f^- of a function as

$$f^+ = max(f, 0)$$

$$\overline{f} = max(-f, 0)$$

Also it was shown that

$$f = f^+ - \overline{f}$$

$$|f| = f^+ + \overline{f}$$

With these notions in mind, we make the following definition.

3.39. Definition. A measurable function f is said to be integrable over E if f^+ and $\overline{f}$ are both integrable over E. In this case we define

$$\int_E f = \int_E f^+ - \int_E \overline{f}$$

3.40. Theorem. Let f and g be integrable over E. Then

(i) The function f+g is integrable over E and

$$\int_E (f+g) = \int_E f + \int_E g$$

(ii) If $f \leq g\ a.e.$, then

$$\int_E f \leq \int_E g$$

(iii) If A and B are disjoint measurable sets contained in E, then

$$\int_{A\cup B} f = \int_A f + \int_B f$$

Proof. By definition, the function $f^+, \overline{f}, g^+, \overline{g}$ are all integrable. If $h = f + g$, then $h = \left(f^+ -, \overline{f}\right) + (g^+ - \overline{g})$ and hence $h = (f^+ + g^+) - (\overline{f} + \overline{g})$. Since $f^+ + g^+$ and $\overline{f} + \overline{g}$ are integrable therefore their difference is also integrable. Thus h is integrable.

We then have

$$\int_E h = \int_E [(f^+ + g^+) - (\overline{f} + \overline{g})]$$

$$= \int_E (f^+ + g^+) - \int_E (\overline{f} + \overline{g})$$

$$= \int_E f^+ + \int_E g^+ - \int_E \overline{f} - \int_E \overline{g}$$

$$= (\int_E f^+ - \int_E \overline{f}) + (\int_E g^+ - \int_E \overline{g})$$

That is, $\int_E (f+g) = \int_E f + \int_E g$

Proof of (ii) follows from part (i) and the fact that the integral of a non-negative integrable function is non-negative.

For (iii) we have $\int_{A\cup B} f = \int f_{\chi_{A\cup B}}$

$$= \int f_{\chi_A} + \int f_{\chi_B} = \int_A f + \int_B f$$

$*$It should be noted that f+g is not defined at points where f = ∞ and g = $-\infty$ and where f = $-\infty$ and g = ∞ . However, the set of such points must have measure zero, since f and g are integrable. Hence the integrability and the value of $\int(f + g)$ is independent of the choice of values in these ambiguous cases.

3.41. Theorem. Let f be a measurable function over E. Then f is integrable over E iff $|f|$ is integrable over E. Moreover, if f is integrable, then

$$\left|\int_E f\right| = \int_E |f|$$

Proof. If f is integrable then both f^+ and f^- are integrable. But $|f| = f^+ + f^-$. Hence integrability of f^+ and f^- implies the integrability of $|f|$.

Moreover, if f is integrable, then since $f(x) \leq |f(x)| = |f|(x)$, the property which states that if $f \leq g$ a.e. , then $\int f \leq \int g$ implies that

$$\int f \leq \int |f| \qquad (i)$$

On the other hand since $-f(x) \leq |f(x)|$, we have

$$-\int f \leq \int |f| \qquad (ii) \quad .$$

From (i) and (ii)

Conversely, suppose f is measurable and suppose $|f|$ is integrable. Since

$$0 \leq f^+(x) \leq |f(x)|$$

It follows that f^+ is integrable. Similarly f^- is also integrable and hence f is integrable.

3.42. Lemma. Let f be integrable . Then given $\varepsilon > 0$ there exist $\delta > 0$ such that $\left|\int_A f\right| < \epsilon$ whenever A is measurable function f we have $= f^+ - f^-$. So by that we have proved already, given > 0 , there exist $\delta_1 > 0$ such that

$$\int_A f^+ < \frac{\varepsilon}{2}.$$

When mA<δ_1. Similarly there exists $\delta_2 > 0$ such that

$$\int_A f^- < \frac{\varepsilon}{2},$$

When mA<δ_2. Thus if mA < δ = min (δ_1, δ_2) , we have

$$\left|\int_A f\right| \leq \int_A |f| = \int_A f^+ + \int_A f^- < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This completes the proof.

3.43. Theorem (**Lebesgue Dominated Convergence Theorem**) Let a sequence $< f_n >$, $n \, \varepsilon \, N$ of measurable functions be dominated by an integrable function g, that is

$$|f_n(x)| \leq g(x)$$

Holds for every $n \, \varepsilon \, N$ and every $x \, \varepsilon \, N$ and let $< f_n >$ converges pointwise to a function f, that is , f(x) = $\lim_{n\to\infty} f_n(x)$ for almost all x in E. Then

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

Proof. Since $|f_n| \leq g$ for every $n \, \varepsilon \, N$ and f(x) = $\lim_{n\to\infty} f_n(x)$, we have $|f| \leq g$. Hence f_n and f are integrable. The function g – f_n is non-negative, therefore by Fatou's Lemma we have

$$\int_E g - \int_E f = \int_E (g - f) \leq \underline{lim} \int_E (g - f_n)$$

$$= \int_E g - \overline{lim} \int_E f_n$$

Whence $\int_E f \geq \overline{\lim} \int_E f_n$

Similarly considering g + f_n we get

$$\int_E f \leq \overline{\lim} \int_E f_n$$

Consequently, we have $\int_E f = \lim \int_E f_n$

SECTION – IV

DIFFERENTIATION AND INTEGRATION

The "fundamental theorem of the integral calculus" is that differentiation and integration are inverse processes. This general principle may be interpreted in two different ways.

If f(x) is integrable, the function

$F(x) = \int_a^x f(t)dt$

is called the indefinite integral of f(x); and the principle asserts that(i) $\acute{F}(x) = f(x)$ (ii)

On the other hand, if F(x) is a given function, and f(x) is defined by (ii), the principle asserts that

$\int_a^x f(t)dt = F(x) - F(a)$ (iii)

The main object of this chapter is to consider in what sense these theorems are true.

From the theory of Riemann integration (ii) follows from (i) if x is a point of continuity of f. For we can choose h_0 so small that $|f(t) - f(x)| < \varepsilon$ for $|t-x| \leq h_0$; and the

$$\frac{F(x+h)-F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h}\{f(t) - f(x)\}dt \leq \varepsilon \qquad (|h| < h_0) \text{ by the mean-value theorem.}$$

This proves (ii).

We shall show that more generally this relation holds almost everywhere. Thus differentiation is the inverse of Lebesgue integration.

The problem of deducing (iii) from (ii) is more difficult and even using Lebesgue integral it is true only for a certain class of functions. We require in the first place that $\acute{F}(x)$should exist at any rate almost everywhere and as we shall see this is not necessarily so. Secondly, if $\acute{F}(x)$ exists we require that it should be integrable.

Differentiation of Monotone Functions

4.1. Definition. Let C be a collection of intervals. Then we say that C covers a set E in the sense of Vitali, if for each $\varepsilon > 0$ and x in E there is an interval $I \in C$ such that $x \in I$ and $l(I) < \varepsilon$.

Now we prove the following lemma which will be utilized in proving a result concerning the differentiation of monotone functions.

4.2. Lemma. (Vitali). Let E be a set of finite outer measure and C a collection of intervals which cover E in the sense of Vitali. Then given $\varepsilon > 0$ there is a finite disjoint collection $\{I_1, \ldots, I_n\}$ of intervals in C such that

$$m^*[E - \cup_{n=1}^{N} I_n] < \varepsilon.$$

Proof. It suffices to prove the lemma in the case that each interval in C is closed, for otherwise we replace each interval by its closure and observe that the set of endpoints of $I_1, I_2, \ldots, I_N$ has measure zero.

Let O be an open set of finite measure containing E. Since C is a Vitali covering of E, we may suppose without loss of generality that each I of C is contained in O. We choose a sequence $< I_n >$ of disjoint intervals of C by induction as follows :

Let I_1 be any interval in C and suppose $I_1, \ldots, I_n$ have already been chosen. Let k_n be the supremum of the lengths of the intervals of C which do not meet any of the intervals $I_1, \ldots, I_n$.

Since each I is contained in O, we have $k_n \leq m\, O < \infty$. Unless, $E \subset \cup_{i=1}^{n} I_I$ we can find I_{n+1} in C with $l(I_{n+1}) > \frac{1}{2} k_n$ and I_{n+1} is disjoint from $I_1, I_2, \ldots, I_n$. Thus we have a sequence $< I_n >$ of disjoint intervals of C, and since $\cup I_n \subset O$, we have $\sum l(I_n) \leq m\, O < \infty$.

Hence we can find an integer N such that $\sum_{N+1}^{\infty} l(I_n) < \frac{\varepsilon}{5}$

Let $R = E - \cup_{n=1}^{N} I_n$.

It remains to prove that m*R < .

Let x be an arbitrary point of R. Since $\cup_{n=1}^{N} I_n$ is a closed set not containing x, we can find an interval I in C which contains x and whose length is so small that I does not meet any of the intervals $I_1, I_2, \ldots, I_N$. If now $I \cap I_i = \emptyset$ for $i \leq N$, we must have $l(I) \leq k_N < 2l(I_{N+1})$. Since $\lim l(I_n) = 0$, the interval I must meet at least one of the intervals I_n. Let n be the smallest integer such that I meets I_n.

We have $n > N$, and $l(I) \leq k_N \leq 2l(I_{N+1})$. Since x is in I, and I has a point in common with I_n, it follows that the distance from x to the midpoint of I_n is at most $l(I) + \frac{1}{2} l(I_N) \leq \frac{5}{2} l(I_{N+1})$.

Let J_m denote the interval which has the same midpoint as I_m and five times the length of I_m. Then we have $x \in J_m$. This proves $R \subset \cup_{N+1}^{\infty} J_n$

Hence $m^* R \leq \sum_{N+1}^{\infty} l(J_n) = 5 \sum_{N+1}^{\infty} l(J_n) < \varepsilon.$

The Four Derivatives of a Function

Whether the differential coefficients

$$\acute{f}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exist or not, the four expressions

$$D^+ f(x) = \overline{\lim_{h \to 0+}} \frac{f(x+h) - f(x)}{h}$$

$$D^- f(x) = \overline{\lim_{h \to 0+}} \frac{f(x) - f(x-h)}{h}$$

$$D_+ f(x) = \underline{\lim}_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$

$$D_- f(x) = \underline{\lim}_{h \to 0+} \frac{f(x) - f(x-h)}{h}$$

always exist. These derivatives are known as Dini Derivatives of the function f.

$D^+ f(x)$ and $D_+ f(x)$ are called upper and lower derivatives on the right and $D^- f(x)$ and $D_- f(x)$ are **called upper and lower derivatives on the left.** Clearly we have $D^+ f(x) \geq D_+ f(x)$ and $D^- f(x) \geq D_- f(x)$. If $D^+ f(x) = D_+ f(x)$, the function f is said to have a **right hand derivative** and if $D^- f(x) = D_- f(x)$, the function is said to have a **left hand derivative.**

If $D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \neq \mp\infty$ we say that f is differentiable at x and define f'(x) to be the common value of the derivatives at x.

4.3.Theorem. Every non-decreasing function f defined on the interval [a, b] is differentiable almost everywhere in [a, b]. The derivative f' is measurable and

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Proof. We shall show first that the points *x* of the open interval (a, b) at which **not** all of the four Dini-derivatives of *f* are equal form a subset of measure zero. It suffices to show that the following four subsets of (a, b) are of measure zero:

$A = \{x \in (a, b) \mid D_- f(x) < D^+ f(x) \}$,

$B = \{x \in (a, b) \mid D_+ f(x) < D^- f(x) \}$,

$C = \{x \in (a, b) \mid D_- f(x) < D^- f(x) \}$,

$D = \{x \in (a, b) \mid D_+ f(x) < D^+ f(x) \}$. To prove $m^* A = 0$, consider the subsets

$A_{u,v} = \{x \in (a, b) \mid D_- f(x) < u < v < D^+ f(x) \}$

of A for all rational numbers u and v satisfying u < v. Since A is the union of this countable family $\{A_{u,v}\}$, it is sufficient to prove $m^* (A_{u,v}) = 0$ for all pairs u, v with u < v.

For this purpose, denote $\alpha = m^* (A_{u,v})$ and let ε be any positive real number. Choose an open set $U \supset A_{u,v}$ with $m^* U < \alpha + \varepsilon$. Set x be any point of $A_{u,v}$. Since $D_- f(x) < u$, there are arbitrary small closed intervals of the form [x-h, x] contained in U such that

$f(x) - f(x-h) < uh$.

Do this for all $x \in A_{u,v}$ and obtain a Vitali cover C of $A_{u,v}$. Then by Vitali covering theorem there is a finite subcollection $\{J_1, J_2,\ldots, J_n\}$of disjoint intervals in C such that

$m^*(A_{u,v} - \cup_{i=1}^{n} J_i) < \varepsilon$

Summing over these n intervals, we obtain

$$\begin{aligned}\sum_{i=1}^{n} f x_i - f(x_i h_i) &< u \sum_{i=1}^{n} h_i \\ &< u\, m^*U \\ &< u(\alpha + \varepsilon)\end{aligned}$$

Suppose that the interiors of the intervals $J_1, J_2,\ldots, J_n$ cover a subset F of $A_{u,v}$. Now since $D^+ f(y) > v$, there are arbitrarily small closed intervals of the form [y, y+k] contained in some of the intervals J_i (i = 1, 2,…, n) such that

$f(y+k) - f(y) > vk$

Do this for all $y \in F$ and obtain a Vitali cover D of F. Then again by Vitali covering lemma we can select a finite subcollection $[K_1, K_2, \ldots, K_m]$ of disjoint intervals in D such that

$m^* [F - \cup_{i=1}^{m} K_i] < \varepsilon$

Since $m^*F > \alpha - \varepsilon$, it follows that the measure of the subset H of F which is covered by the intervals is greater than $\alpha - 2\varepsilon$. Summing over these intervals and keeping in mind that each K_i is contained in a J_n, we have

$$\sum_{i=1}^{n}\{f x_i - f(x_i - h_i)\} \geq \sum_{i=1}^{m} [f(y_i + k_i) - f y_i]$$
$$> v \sum_{i=1}^{n} k_i$$
$$> v(\alpha - 2\varepsilon)$$

So that

$$v(\alpha - 2\varepsilon) < u(\alpha - \varepsilon)$$

Since this is true for every $\varepsilon > 0$, we must have $v\,\alpha < u\,\alpha$. Since $u < v$, this implies that $\alpha = 0$. Hence $m^*A = 0$. Similarly, we can prove that $m^*B = 0$, $m^*C = 0$ and $m^*D = 0$. This shows that

$$g(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is defined almost everywhere and that f is differentiable whenever g is finite.

If we put

$$g_n(x) = n[f(x+\tfrac{1}{n}) - f(x)] \qquad \text{for } x \in [a,b] ,$$

where we re-define $f(x) = f(b)$ for $x \geq b$. Then $g_n(x) \to g(x)$ for almost all x and so g is measurable since every g_n is measurable. Since f is non-decreasing, we have $g_n \geq 0$. Hence, by

Fatou's Lemma

$$\int_a^b g \leq \underline{lim} \int_a^b g_n = \underline{lim}\; n \int_a^b [f\left(x + \frac{1}{n}\right) - f(x)]dx$$

$$= \underline{lim}\; n \int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)dx - \int_a^b f(x)dx$$

$$= \underline{lim}\; n \int_a^b [f(x) + \int_b^{b+\frac{1}{n}} f(x)dx - \int_a^{a+\frac{1}{n}} f(x)dx - \int_a^b f(x)dx$$

$$= \underline{lim}\; n \int_a^b [f\left(x + \frac{1}{n}\right) - f(x)]dx$$

$$\leq f(b)-f(a)$$

(Use of $f(x) = f(b)$ for $x \geq b$ for first interval and f non-decreasing in the 2^{nd} integral).

This shows that g is integrable and hence finite almost everywhere. Thus f is differentiable almost everywhere and g(x) = $\acute{f}(x)$ almost everywhere. This proves the theorem.

Functions of Bounded Variation

Let f be a real-valued function defined on the interval [a,b] and let $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be any partition of [a,b].

By the variation of f over the partition $P = \{x_0, x_1, \ldots, x_n\}$ of [a,b], we mean the real number $V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$

and then

$$V_a^b(f) = \sup \{V(f,P) \text{ for all possible partitions } P \text{ of } [a,b]\}$$
$$= sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

is called the total variation of f over the interval [a,b]. If $V_a^b(f) < \infty$ then we say that f is a function of bounded variation and we write f ∈ BV.

4.4. Lemma. Every non-decreasing function f defined on the interval [a,b] is of bounded variation with total variation

$$V_a^b(f) = f(b) - f(a).$$

Prof. For every partition $P = [x_0, x_1, \ldots, x_n\}$ of [a,b], we have

$$V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$
$$= f(b) - f(a)$$

This implies the lemma.

4.5.Theorem. (Jordan Decomposition Theorem). A function f: [a,b] → **R** is of bounded variation if and only if it is the difference of two non-decreasing functions.

Proof. Let f = g-h on [a,b] with g and h increasing. Then for any, subdivision we have

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n} |h(x_i) - h(x_{i-1})|$$
$$= g(b) - g(a) + h(b) - h(a)$$

Hence,

$$V_a^b(f) \le g(b) - g(a) + h(b) - h(a),$$

which proves that f is of bounded variations.

On the other hand, let f be of bounded variation. Define two functions g, h : [a, b] →R by taking

$$g(x) = V_a^x(f), \quad h(x) = V_a^x(f) - f(x) \quad \text{for every } x \in [a, b]. \text{ Then } f(x) = g(x) - h(x).$$

The function g is clearly non-decreasing. On the other hand, for any two real numbers x and y in [a, b] with $x \le y$, we have

$$h(y) - h(x) = [V_a^x(f) - f(y)] - [V_a^x(f) - f(x)]$$

$= V_x^y(f) - [f(y) - f(x)]$

$\geq V_x^y(f) - V_x^y(f) = 0$

Hence h is also non-decreasing. This completes the proof of the theorem.

4.6. Examples. (1) If f is monotonic on [a,b], then f is of bounded variation on [a, b] and V(f) = |f(b)-f(a)| , where V(f) is the total variation.

(2) If $\acute{f}(x)$ exists and is bounded on [a, b], then f is of bounded variation. For if $|\acute{f}(x)| \leq M$ we have

$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n} M|x_i - x_{i-1}| = M(b-a)$

no matter which partition we choose.

(3) *f* may be continuous without being of bounded variation. Consider

$$f(x) = \begin{cases} x \sin\frac{\pi}{x} & (0 < x \leq 2) \\ 0 & (x = 0) \end{cases}$$

Let us choose the partition which consists of the points

$$0, \frac{2}{2n-1}, \frac{2}{2n-3}, \frac{2}{2n-5}, \ldots\ldots \frac{2}{5}, \frac{2}{3}, 2$$

Then the sum in the total variation is

$$\left(2 + \frac{2}{3}\right) + \left(\frac{2}{3} + \frac{2}{5}\right) + \ldots.. + \left(\frac{2}{2n-3} + \frac{2}{2n-1}\right) + \frac{2}{2n-1} > \frac{1}{2} + \frac{1}{3} + \ldots.. + \frac{1}{n}$$

and this can be made arbitrarily large by taking n large enough, since $\sum \frac{1}{n}$ diverges.

(4). Since $|f(x) - f(a)| \leq V(f)$ for every x on [a,b] it is clear that every function of bounded variation is bounded.

The Differentiation of an Integral

Let f be integrable over [a,b] and let

$F(x) = \int_a^x f(t)dt$

If f is positive, h > 0 , then we see that

$F(x+h) - F(x) = \int_x^{x+h} f(t)dt$

Hence, integral of a positive function is non-decreasing.

We shall show first that F is a function of bounded variation. Then, being function of bounded variation, it will have a finite differential coefficient F' almost everywhere. Our object is to prove that $\acute{F}(x) = f(x)$ almost everywhere in [a,b]. We prove the following lemma :

4.7. Lemma. If f is integrable on [a,b], then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is a continuous function of bounded variation on [a,b].

Proof. We first prove continuity of F. Let x_0 be an arbitrary point of [a,b]. Then

$|F(x) - F(x_0)| = |\int_{x_0}^{x} f(t)dt\ |$

$$\leq \int_{x_0}^{x} |f(t)|dt$$

Now the integrability of *f* implies integrability of |f| over [a,b]. Therefore, given $\varepsilon > 0$ there is a $\delta > 0$ such that for every measurable set A $\subset$[a,b] with measure less than δ, we have $\int_A |f| < \varepsilon$.Hence

$|F(x)-F(x_0)| < \varepsilon$ whenever $|x-x_0| < \delta_1$

and so f is continuous.

To show that F is of bounded variation, let a = $x_0 < x_1 < \ldots < x_n$ = b be any partition of [a,b]. Then

$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} |\int_a^{x_i} f(t)dt - \int_a^{x_{i-1}} f(t)dt\ |$

$= \sum_{i=1}^{n} |\int_{x_{i-1}}^{x_i} f(t)dt\ |$

$\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|dt$

$= \int_a^b |f(t)|dt$

Thus $\quad V_a^b(f) \leq \int_a^b |f(t)|dt < \infty$

Hence F is of bounded variation.

4.8. Lemma. If f is integrable on [a, b] and

$$\int_a^x f(t)dt = 0$$

for all x ∈ [a,b], then f = 0 almost everywhere in [a,b].

Proof. Suppose f > 0 on a set E of positive measure. Then there is a closed set F ⊂ E with m F > 0. Let O be the open set such that

O = (a, b) – F

Then either $\int_a^b f \neq 0$ or else

$$0 = \int_a^b f = \int_A f + \int_O f$$

$$= \int_F f + \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(t)dt \qquad (1)$$

because O is the union of a countable collection $\{(a_n, b_n)\}$ of open intervals.

But, for each n,

$$\int_{a_n}^{b_n} f(t)dt = \int_a^{b_n} f(t)dt - \int_a^{a_n} f(t)dt$$

$= F(b_n) - F(a_n) = 0$ (by hypothesis) Therfore, from (1), we have

$$\int_F f = 0$$

But since f > 0 on F and mF > 0, we have $\int_F f > 0$.

We thus arrive at a contradiction. Hence f = 0 almost everywhere.

4.9. Lemma. If f is bounded and measurable on [a, b] and

$$F(x) = \int_F^x f(t)dt + F(a),$$

then F'(x) = f(x) for almost all x in [a,b]. $f_n(x) = \frac{1}{h}[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt]$

Proof. We know that an integral is of bounded variation over [a,b] and so F'(x) exists for almost all x in [a,b]. Let $|f| \leq K$. We set

$$f_n(x) = \frac{F(x+h)-F(x)}{h}$$

with $h = \frac{1}{n}$. Then we have

$$f_n(x) = \frac{1}{h}[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt]$$

$$= \frac{1}{h}[\int_x^{x+h} f(t)dt$$

implies $|f_n(x)| = |\frac{1}{h}[\int_x^{x+h} f(t)dt| \leq \frac{1}{h}\int_x^{x+h} |f(t)|dt$

$$\leq \frac{1}{h} \int_x^{x+h} K\, dt$$

$= \frac{K}{h}.\, h = K$

Moreover,

$$f_n(x) \to F'(x)$$

Hence by the theorem of bounded convergence, we have

$$\int_a^c F'(x)dx = \lim \int_a^c f_n(x)dx = \lim_{h\to 0} \frac{1}{h}\int_a^c [F(x+h) - F(x)]dx$$

$$= \lim_{h\to 0} [\frac{1}{h}\int_{a+h}^{c+h} F(x)dx - \frac{1}{h}\int_a^c F(x)\, dx]$$

$$= \lim_{h\to 0} [\frac{1}{h}\int_c^{c+h} F(x)dx - \frac{1}{h}\int_a^{a+h} F(x)\, dx]$$

$$= F(c) - F(a)$$

$$= \int_a^c f(x)dx$$

Hence,

$$\int_a^c [F'(x) - f(x)]dx = 0$$

For all $c \in [a,b]$, and so

$$F'(x) = f(x) \text{ a.e.}$$

By using pervious lemma.

Now we extend the above lemma to unbounded functions.

4.10. Theorem. Let f be an integrable function on [a,b] and suppose that

$$F(x) = F(a) + \int_a^x f(x)dt$$

Then $F'(x) = f(x)$ for almost all in x in [a,b].

Proof. Without loss of generality we may assume that $f \geq 0$ (or we may write "From the definition of integral it is sufficient to prove the theorem when $f \geq 0$).

Let f_n be defined by $f_n(x) = f(x)$ if $f(x) \leq n$ and $f_n(x) = n$ if $f(x) > n$. Then $f - f_n \geq 0$ and so

$$G_n(x) = \int_a^x (f - f_n)$$

is an increasing function of x, which must have a derivative almost everywhere and this derivative will be non-negative. Also by the above lemma, since f_n is bounded (by n), we have

$$\frac{d}{dx}\left(\int_a^x f_n\right) = f_n(x) \text{ a.e.}$$

Therefore,

$$F'(x) = \frac{d}{dx}\left(\int_a^x f\right) = \frac{d}{dx}\left(G_n + \int_a^x f_n\right)$$

$$= \frac{d}{dx}\left(G_n + \frac{d}{dx}\left(\int_a^x f_n\right) \geq f_n(x) \quad \text{a.e.} \qquad \text{(using (i))}$$

Since n is arbitrary, making $n \to \infty$ we see that

$F'(x) \geq f(x)$ a.e.

Consequently,

$$\int_a^b F'(x)dx \geq \int_a^b f(x)dx = F(b) - F(a) \qquad \text{(using the hypothesis of the theorem)}$$

Also since F(x) is an increasing real valued function on the interval [a,b], we have

$$\int_a^b F'(x)dx \leq F(b) - F(A) = \int_a^b f(x)dx$$

Hence

$$\int_a^b F'(x)dx = F(b) - F(A) = \int_a^b f(x)dx$$

implies $\int_a^b [F'(x) - f(x)]dx = 0$

Since $F'(x) - f(x) \geq 0$, this implies that $F'(x) - f(x) = 0$ a.e. and so $F'(x) = f(x)$ a.e.

Absolute Continuity

4.11. Definition. A real-valued function f defined on [a,b] is said to be **absolutely continuous** on [a,b] if, given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x_i') - F(x_i)| < \varepsilon$$

for every finite collection $\{(x_i, x_i')\}$ of non-overlapping intervals with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta$$

An absolutely continuous function is continuous, since we can take the above sum to consist of one term only. Moreover, if

$$F(x) = \int_a^x f(t)dt$$

Then

$$\sum_{i=1}^{n} |f(x_i') - F(x_i)| = \sum_{i=1}^{n} \left| \int_a^{x_i'} f(t)dt - \int_a^{x_i} f(t)dt \right|$$

$$= \sum_{i=1}^{n} \left| \int_{x_i}^{x_i'} f(t)dt \right|$$

$$\leq \sum_{i=1}^{n} \int_{x_i}^{x_i'} |f(t)|dt = \int_E |f(t)|dt,$$

where E is the set of intervals (x, x_i') $\leq$ $\to 0$ as $\sum_{i=1}^{n} |x_i' - x_i| \to 0$

The last step being the consequence of the result.

"Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that for every measurable set $E \subset [a, b]$ with

$mE < \delta$, we have $\int_A |f| < \varepsilon$".

Hence every indefinite integral is absolutely continuous.

4.12. Lemma. If f is absolutely continuous on [a,b], then it is of bounded variation on [a,b].

Proof. Let δ be a positive real number which satisfies the condition in the definition for $\varepsilon = 1$. Select a natural number

$$n > \frac{b-a}{\delta}$$

Consider the partition $\pi = \{x_0, x_1, \ldots, x_n\}$ of [a,b] defined by

$$x_i = x_0 + \frac{i(b-a)}{n}$$

for every $i = 0, 1, \ldots, n$. Since $|x_i - x_{i-1}| < \delta$, it follows that

$$V_{x_{i-1}}^{x_i}(f) < 1.$$

This implies

$$V_a^b(f) = \sum_{i=1}^{n} V_{x_{i-1}}^{x_i}(f) < n$$

Hence f is of bounded variation.

4.13. Corollary. If f is absolutely continuous, then f has a derivative almost everywhere.

Proof: Since f is absolutely continuous, then by above theorem, f is of bounded variation and hence f has a derivative almost everywhere (by theorem 4.3).

4.14. Lemma. If f is absolutely continuous on [a,b] and f'(x) = 0 a.e., then f is constant.

Proof. We wish to show that f(a) = f(c) for any $c \in [a,b]$.

Let $E \subset (a,c)$ be the set of measure c-a in which f'(x) = 0, and let ε and η be arbitrary positive numbers. To each x in E there is an arbitrarily small interval [x, x+h] contained in [a,c] such that

$|f(x+h) - f(x)| < \eta h$

By Vitali Lemma we can find a finite collection $\{[x_k, y_k]\}$ of non-overlapping intervals of this sort which cover all of E except for a set of measure less than δ, where δ is the positive number corresponding to ε in the definition of the absolute continuity of f . If we label the x_k so that $x_k \le x_{k+1}$, we have (or if we order these intervals so that)

$$a = y_0 \le x_1 < y_1 \le x_2 < \ldots . < y_n \le x_{n+1} = c$$

and

$$f\left(\sum_{k=0}^{n} |x_{k+1} - y_k|\right) < \delta$$

Now, $\sum_{k=0}^{n} |(y_k) - f(x_k)| < \eta \sum_{k=1}^{n} |y_k - x_k| < \eta(c-a)$

by the way to intervals $\{[x_k, y_k]\}$ were constructed, and

$$\sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| < \varepsilon$$

by the absolute continuity of *f*. Thus

$$|f(c) - f(a)| = \sum_{k=0}^{n} [\,|f(x_{k+1}) - f(y_k)|\,] + \sum_{k=1}^{n} [\,|f(y_k) - f(x_k)]\,| \le \varepsilon + \eta(c-a)$$ Since ε and η are arbitrary positive numbers, f(c) − f(a) = 0 and so f(c) = f(a).

Hence f is constant.

4.15. Theorem. A function F is an indefinite integral if and only if it is absolutely continuous.

Proof. Let function F is an indefinite integral then

$F(x) = \int_a^x f(x)dt$

where f is integrable on [a, b]

Now f is integrable on [a, b]

⇨ $|f|$ is integrable on [a, b]. Then for given $\varepsilon > 0$, there is a $\delta > 0$ such that for every measurable set A contained in [a, b] with m(A) < δ, we have $\int_A |f| < \varepsilon$

Let $\{(x_i, x_i')\}_{i=1}^n$ be any finite collection of pairwise disjoint interval in [a, b] such that $\sum_{i=1}^n |x_i - x'_i| < \delta$

Let A = $\bigcup_{i=1}^n (x_i, x'_i)$

Then m(A) = $\sum_{i=1}^n |x_i - x'_i| < \delta$

Therefore we have $\int_A |f| < \varepsilon$

i.e., $\int_{\bigcup_{i=1}^n (x_i, x'_i)} |f| < \varepsilon$

=> $\sum_{i=1}^n \int_{x_i}^{x'_i} |f| < \varepsilon$ …(1)

Consider $\sum_{i=1}^n |F(x'_i) - F(x_i)| = \sum_{i=1}^n \left|\int_a^{x'_i} f(t)dt - \int_a^{x_i} f(t)dt\right|$

$$= \sum_{i=1}^n \left|\int_{x_i}^{x'_i} f(t)dt\right|$$

$$\leq \sum_{i=1}^n \int_{x_i}^{x'_i} |f(t)|\, d(t)$$

$$< \varepsilon [by\ (1)]$$

Conversely, Suppose F is absolutely continuous on [a,b]. Then F is of bounded variation and we may write

$F(x) = F_1(x) - F_2(x)$,

where the functions F_i are monotone increasing. Hence F'(x) exists almost everywhere and $|F'(x)| \leq F_1'(x) + F_2'(x)$

Thus $\int |F'(x)|dx \leq F_1(b) + F_2(b) - F_1(a) - F_2(a)$

and F'(x) is integrable. Let

$$G(x) = \int_a^x F'(t)dt$$

Then G is absolutely continuous and so is the function f = F−G. But by the above lemma since

f'(x) = F'(x)– G'(x)= 0 a.e., we have f to be a constant function. That is,

F(x) − G(x) = A (constant)

or

$F(x) = \int_a^x F'(t)dt = A$

or

$F(x) = \int_a^x F'(t)dt + A$

Taking x = a, we have A = F(a) and so

$F(x) = \int_a^x F'(t)dt + F(a)$

Thus F(x) is indefinite integral of F'(x).